3.1 $\underset{\sim}{x}_i \sim N_2(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$ independent

$$\underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$H_0: \mu_1 = \mu_2$, $H_1: \mu_1 \neq \mu_2$

$$y_i = x_{i1} - x_{i2} \sim N(\mu_1 - \mu_2, \sigma_y^2), \quad \sigma_y^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

$y_i \sim N(0, \sigma_y^2)$ when H_0 is true

$$T^2 = n\bar{y}\, S_y^{-1} \bar{y} = \frac{n\bar{y}^2}{s_y^2} = \left(\frac{\sqrt{n}\,\bar{y}}{s_y}\right)^2 \sim T^2(1, n-1)$$

$$T^2(1, n-1) = \frac{(n-1)1}{(n-1)-1+1} F(1, (n-1)-1+1) = F(1, n-1)$$

$$\left(\frac{\sqrt{n}\,\bar{y}}{s_y}\right)^2 \sim F(1, n-1) \Rightarrow \frac{\sqrt{n}\,\bar{y}}{s_y} \sim t(n-1)$$

3.2 $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$ independent

A $(d-1) \times d$, rank $A = d-1$

$$A \underset{\sim}{1}_d = \underset{\sim}{0} \iff \underset{\sim}{1}_d^T A^T = \underset{\sim}{0}^T$$

$$A^T (ASA^T)^{-1} A = S^{-1} - S^{-1}\underset{\sim}{1}_d (\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d)^{-1} \underset{\sim}{1}_d^T S^{-1} \quad \text{cf. B 3.5}$$

$$= S^{-1} - \frac{S^{-1}\underset{\sim}{1}_d \underset{\sim}{1}_d^T S^{-1}}{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d}$$

$$\Rightarrow n\underset{\sim}{\bar{x}}^T A^T (ASA^T)^{-1} A \underset{\sim}{\bar{x}} = n\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{\bar{x}} - \frac{n\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{1}_d \underset{\sim}{1}_d^T S^{-1} \underset{\sim}{\bar{x}}}{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d}$$

$$= n\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{\bar{x}} - \frac{n(\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{1}_d)^2}{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d}$$

3.3

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $\quad i = 1, \dots, n$, $\quad$ independent

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$$

$$Q_H = \sum_i \sum_j (\bar{x}_{.j} - \bar{x}_{..})^2 = n \sum_j (\bar{x}_{.j} - \bar{x}_{..})^2$$

$$Q_E = \sum_i \sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2$$

H_0: $\mu_1 = \mu_2 = \dots = \mu_d \; (= \mu) \iff \underset{\sim}{\mu} = \mu \underset{\sim}{1}_d$

H_1: not all the μ_j's are equal

$$\mathrm{Cov}(\bar{x}_{.k} - \bar{x}_{..},\; x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})$$

$$= \mathrm{Cov}\Big(\frac{1}{n}\sum_l x_{lk} - \frac{1}{nd}\sum_l \sum_k x_{lk},$$
$$x_{ij} - \frac{1}{d}\sum_j x_{ij} - \frac{1}{n}\sum_i x_{ij} + \frac{1}{nd}\sum_i\sum_j x_{ij}\Big).$$

$$= \frac{1}{n}\mathrm{Cov}(x_{ik}, x_{ij}) - \frac{1}{nd}\sum_j \mathrm{Cov}(x_{ik}, x_{ij})$$
$$- \frac{1}{n^2}\sum_i \mathrm{Cov}(x_{ik}, x_{ij}) + \frac{1}{n^2 d}\sum_i\sum_j \mathrm{Cov}(x_{ik}, x_{ij})$$
$$- \frac{1}{nd}\sum_k \mathrm{Cov}(x_{ik}, x_{ij}) + \frac{1}{nd^2}\sum_j\sum_k \mathrm{Cov}(x_{ik}, x_{ij})$$
$$+ \frac{1}{n^2 d}\sum_i\sum_k \mathrm{Cov}(x_{ik}, x_{ij}) - \frac{1}{n^2 d^2}\sum_i\sum_j\sum_k \mathrm{Cov}(x_{ik}, x_{ij})$$

$$= \frac{1}{n}\mathrm{Cov}(x_{ik}, x_{ij}) - \frac{1}{nd}\sum_j \mathrm{Cov}(x_{ik}, x_{ij})$$
$$- \frac{1}{n^2} n\, \mathrm{Cov}(x_{ik}, x_{ij}) + \frac{1}{n^2 d} n \sum_k \mathrm{Cov}(x_{ik}, x_{ij})$$
$$- \frac{1}{nd}\sum_k \mathrm{Cov}(x_{ik}, x_{ij}) + \frac{1}{nd^2} d \sum_j \mathrm{Cov}(x_{ik}, x_{ij})$$
$$+ \frac{1}{n^2 d} n \sum_k \mathrm{Cov}(x_{ik}, x_{ij}) - \frac{1}{n^2 d^2} nd \sum_k \mathrm{Cov}(x_{ik}, x_{ij})$$

$$= 0$$

$$\mathrm{Cov}\Big(\bar{\underset{\sim}{x}} - \bar{x}_{..}\underset{\sim}{1}_d,\; \underset{\sim}{x}_i - \frac{1}{d}\underset{\sim}{1}_d \underset{\sim}{1}_d^T \underset{\sim}{x}_i - \bar{\underset{\sim}{x}} + \bar{x}_{..}\underset{\sim}{1}_d\Big) = 0$$

Let $\underset{\sim}{x} = [\underset{\sim}{x}_1^T \; \underset{\sim}{x}_2^T \; \dots \; \underset{\sim}{x}_n^T]^T$ and $\underset{\sim}{\theta} = [\underset{\sim}{\mu}^T \; \underset{\sim}{\mu}^T \; \dots \; \underset{\sim}{\mu}^T]^T$

$$\left.\begin{array}{l} \underset{\sim}{x} \sim N_{nd}(\underset{\sim}{\theta}, I_n \otimes \Sigma) \\ \underset{\sim}{\bar{x}} - \bar{x}_{..} \underset{\sim}{1}_d = A_1 \underset{\sim}{x} \\ \underset{\sim}{x}_i - \frac{1}{d} \underset{\sim}{1}_d \underset{\sim}{1}_d^T \underset{\sim}{x}_i - \underset{\sim}{\bar{x}} + \bar{x}_{..} \underset{\sim}{1}_d = A_2 \underset{\sim}{x} \\ \mathrm{Cov}(A_1 \underset{\sim}{x}, A_2 \underset{\sim}{x}) = 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \underset{\sim}{\bar{x}} - \bar{x}_{..} \underset{\sim}{1}_d \\ \underset{\sim}{x}_i - \frac{1}{d} \underset{\sim}{1}_d \underset{\sim}{1}_d^T \underset{\sim}{x}_i - \underset{\sim}{\bar{x}} + \bar{x}_{..} \underset{\sim}{1}_d \end{array}\right\} \text{indep., cf. th. 2.1 (v)} \Rightarrow$$

$$\left.\begin{array}{l} \sum_j (\bar{x}_{.j} - \bar{x}_{..})^2 \\ \sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2 \end{array}\right\} \text{indep.} \Rightarrow$$

$$\left.\begin{array}{l} Q_H \\ Q_E \end{array}\right\} \text{indep.}$$

$$\underset{\sim}{\bar{z}} = \sqrt{\frac{n}{1-\rho}}(\underset{\sim}{\bar{x}} - \mu \underset{\sim}{1}_d) \sim N_d\Big(\underbrace{\sqrt{\frac{n}{1-\rho}}(\underset{\sim}{\mu} - \mu \underset{\sim}{1}_d)}_{= \underset{\sim}{0},\ H_0 \text{ true}}, \underbrace{\frac{n}{1-\rho}\frac{1}{n}\Sigma}_{= \frac{1}{1-\rho}\Sigma}\Big)$$

$$\begin{aligned} Q_H &= n \sum_j (\bar{x}_{.j} - \bar{x}_{..})^2 \\ &= n \sum_j (\bar{x}_{.j} - \mu - \bar{x}_{..} + \mu)^2 \\ &= n \frac{1-\rho}{n} \sum_j (\bar{z}_{.j} - \bar{z}_{..})^2 \\ &= (1-\rho)\, \underset{\sim}{\bar{z}}^T (I_d - \tfrac{1}{d} \underset{\sim}{1}_d \underset{\sim}{1}_d^T) \underset{\sim}{\bar{z}} \qquad \text{cf. ex. 2.8} \end{aligned}$$

$$\frac{1}{1-\rho} Q_H = \underset{\sim}{\bar{z}}^T A \underset{\sim}{\bar{z}}, \qquad A = I_d - \frac{1}{d} \underset{\sim}{1}_d \underset{\sim}{1}_d^T$$

A is symmetric and idempotent

rank A = tr A = $d-1$

$$\frac{1}{1-\rho} Q_H \sim \sigma^2 \chi^2(d-1) \quad \text{when } H_0 \text{ is true, cf. A 6.5}$$

$$\mathbf{z} = \sqrt{\frac{1}{1-\rho}}\,(\mathbf{x} - \mu \mathbf{1}_{nd}) \sim N_{nd}\Big(\underbrace{\sqrt{\frac{1}{1-\rho}}\,(\boldsymbol{\theta} - \mu \mathbf{1}_{nd})}_{=\,\mathbf{0},\ H_0 \text{ true}},\ \underbrace{\frac{1}{1-\rho}\, I_n \otimes \Sigma}_{=\,\Sigma_0} \Big)$$

$$\begin{aligned}
Q_E &= \sum_i \sum_j (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2 \\
&= \sum_i \sum_j (x_{ij} - \mu - \bar{x}_{i.} + \mu - \bar{x}_{.j} + \mu + \bar{x}_{..} - \mu)^2 \\
&= (1-\rho) \sum_i \sum_j (z_{ij} - \bar{z}_{i.} - \bar{z}_{.j} + \bar{z}_{..})^2
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1-\rho} Q_E &= \Big(\mathbf{z} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T \mathbf{z} - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \mathbf{z} + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T \mathbf{z} \Big)^T \\
&\qquad \Big(\mathbf{z} - \tfrac{1}{d} I_n \otimes \mathbf{1}_n \mathbf{1}_n^T \mathbf{z} - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d \mathbf{z} + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T \mathbf{z} \Big) \\
&= \mathbf{z}^T \Big(I_{nd} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T \Big)^T \\
&\qquad \Big(I_{nd} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T \Big) \mathbf{z} \\
&= \mathbf{z}^T A_0 \mathbf{z}
\end{aligned}$$

$$\begin{aligned}
A_0 = {} & I_{nd} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T \\
& - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T + \tfrac{1}{d^2} I_n \otimes d\, \mathbf{1}_d \mathbf{1}_d^T + \tfrac{1}{nd} \mathbf{1}_n \mathbf{1}_n^T \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{nd^2} d\, \mathbf{1}_{nd} \mathbf{1}_{nd}^T \\
& - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \tfrac{1}{nd} \mathbf{1}_n \mathbf{1}_n^T \otimes \mathbf{1}_d \mathbf{1}_d^T + \tfrac{1}{n^2} n\, \mathbf{1}_n \mathbf{1}_n^T \otimes I_d - \tfrac{1}{n^2 d} n\, \mathbf{1}_{nd} \mathbf{1}_{nd}^T \\
& + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T - \tfrac{1}{nd^2} d\, \mathbf{1}_{nd} \mathbf{1}_{nd}^T - \tfrac{1}{n^2 d} n\, \mathbf{1}_{nd} \mathbf{1}_{nd}^T + \tfrac{1}{n^2 d^2} nd\, \mathbf{1}_{nd} \mathbf{1}_{nd}^T \\
= {} & I_{nd} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T
\end{aligned}$$

A_0 is symmetric and idempotent

$$\begin{aligned}
\operatorname{rank} A_0 = \operatorname{tr} A_0 &= nd - d - n + 1 \\
&= (n-1)\,d - (n-1) \\
&= (n-1)(d-1)
\end{aligned}$$

$$\frac{1}{1-\rho} Q_E \sim \sigma^2 \chi^2\big((n-1)(d-1)\big) \quad \text{when } H_0 \text{ is true,}$$

cf. A 6.5

$$\left.\begin{array}{l} \dfrac{Q_H}{\sigma^2(1-\rho)} \sim \chi^2(d-1) \quad \text{when } H_0 \text{ is true} \\[2ex] \dfrac{Q_E}{\sigma^2(1-\rho)} \sim \chi^2\big((n-1)(d-1)\big) \quad \text{when } H_0 \text{ is true} \\[2ex] Q_H \text{ og } Q_E \text{ uafhængige} \end{array}\right\} \Rightarrow$$

$$F = \frac{\dfrac{\dfrac{Q_H}{\sigma^2(1-\rho)}}{d-1}}{\dfrac{\dfrac{Q_E}{\sigma^2(1-\rho)}}{(n-1)(d-1)}} = \frac{(n-1)\,Q_H}{Q_E}$$

$$\sim F\big(d-1,\ (n-1)(d-1)\big) \quad \text{when } H_0 \text{ is true},$$

cf. definition of the F distribution

$$EQ_H = E\left[(1-\rho)\,\underset{\sim}{\bar{z}}^T A\, \underset{\sim}{\bar{z}}\right]$$

$$= (1-\rho)\Big(\operatorname{tr}\big(A\,\frac{\Sigma}{1-\rho}\big)$$

$$+ \frac{n}{1-\rho}\,(\underset{\sim}{\mu} - \mu\,\underset{\sim}{1}_d)^T A\,(\underset{\sim}{\mu} - \mu\,\underset{\sim}{1}_d)\Big)$$

$$= \operatorname{tr}(A\Sigma) + n(\underset{\sim}{\mu} - \mu \underset{\sim}{1}_d)^T A (\underset{\sim}{\mu} - \mu \underset{\sim}{1}_d)$$

$A\Sigma = \sigma^2(1-\rho)A$, as

$$A\Sigma \neq \sigma^2(1-\rho)A \Rightarrow A\frac{\Sigma}{1-\rho}A \neq \sigma^2 A^2$$

$$\Leftrightarrow A\frac{\Sigma}{1-\rho}A \neq \sigma^2 A \quad \text{inconsistency *}$$

$$\operatorname{tr}(A\Sigma) = \sigma^2(1-\rho)\operatorname{tr}A = \sigma^2(1-\rho)(d-1)$$

$$A(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_d) = (I_d - \tfrac{1}{d}\underset{\sim}{1}_d\underset{\sim}{1}_d^T)(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_d)$$

$$= \underset{\sim}{\mu} - \mu\underset{\sim}{1}_d - \tfrac{1}{d}\underset{\sim}{1}_d\sum_j \mu_j + \tfrac{\mu}{d}\underset{\sim}{1}_d d = \underset{\sim}{\mu} - \bar{\mu}\underset{\sim}{1}_d$$

$$(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_d)^T A (\underset{\sim}{\mu} - \mu\underset{\sim}{1}_d) = (\underset{\sim}{\mu} - \mu\underset{\sim}{1}_d)^T(\underset{\sim}{\mu} - \bar{\mu}\underset{\sim}{1}_d)$$

$$= \underset{\sim}{\mu}^T\underset{\sim}{\mu} - \bar{\mu}\sum_j \mu_j - \mu\sum_i \mu_i + \mu\bar{\mu}d$$

$$= \sum_j \mu_j^2 - d\bar{\mu}^2 - d\mu\bar{\mu} + d\mu\bar{\mu} = \sum_j(\mu_j - \bar{\mu})^2$$

$$E\,Q_H = (d-1)\sigma^2(1-\rho) + n\sum_j(\mu_j - \bar{\mu})^2$$

$$E[(n-1)Q_H] = (n-1)(d-1)\sigma^2(1-\rho) + (n-1)n\sum_j(\mu_j-\bar{\mu})^2$$

$$= (n-1)(d-1)\sigma^2(1-\rho) \quad \text{when } H_0 \text{ is true}$$

$$E\,Q_E = E\left[(1-\rho)\underset{\sim}{z}^T A_0 \underset{\sim}{z}\right]$$

$$= (1-\rho)\left(\operatorname{tr}(A_0\Sigma_0) + \frac{1}{1-\rho}(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_{nd})^T A_0(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_{nd})\right)$$

$$= (1-\rho)\operatorname{tr}(A_0\Sigma_0) + (\underset{\sim}{\mu} - \mu\underset{\sim}{1}_{nd})^T A_0(\underset{\sim}{\mu} - \mu\underset{\sim}{1}_{nd})$$

$A_0\Sigma_0 = \sigma^2 A_0$, as

$$A_0\Sigma_0 \neq \sigma^2 A_0 \Rightarrow A_0\Sigma_0A_0 \neq \sigma^2 A_0^2 \Leftrightarrow A_0\Sigma_0A_0 \neq \sigma^2 A_0$$

inconsistency *

$$\operatorname{tr}(A_0\Sigma_0) = \sigma^2\operatorname{tr}A_0 = \sigma^2(n-1)(d-1)$$

* cf. exercise 1.9 modified

$$A_0(\boldsymbol{\mu} - \mu \mathbf{1}_{nd}) = \left(I_{nd} - \tfrac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \tfrac{1}{n} \mathbf{1}_n \mathbf{1}_n^T + \tfrac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T\right)(\boldsymbol{\mu} - \mu \mathbf{1}_{nd})$$

$$= \boldsymbol{\mu} - \mu \mathbf{1}_{nd} - \tfrac{1}{d}\left(\textstyle\sum_j \mu_j\right) \mathbf{1}_{nd} + \tfrac{\mu}{d}\, d\, \mathbf{1}_{nd} - \tfrac{1}{n}\, n\, \boldsymbol{\mu}$$

$$+ \tfrac{\mu}{n}\, n\, \mathbf{1}_{nd} + \tfrac{1}{nd}\, n \left(\textstyle\sum_j \mu_j\right) \mathbf{1}_{nd} - \tfrac{\mu}{nd} \mathbf{1}_{nd}\, nd$$

$$= \boldsymbol{\mu} - \mu \mathbf{1}_{nd} - \bar{\mu} \mathbf{1}_{nd} + \mu \mathbf{1}_{nd} - \boldsymbol{\mu} + \mu \mathbf{1}_{nd} + \bar{\mu} \mathbf{1}_{nd} - \mu \mathbf{1}_{nd}$$

$$= \mathbf{0}$$

$$(\boldsymbol{\mu} - \mu \mathbf{1}_{nd})^T A_0 (\boldsymbol{\mu} - \mu \mathbf{1}_{nd}) = 0$$

$$E\, Q_E = (n-1)(d-1)\,\sigma^2 (1-\rho) \qquad (H_0 \text{ true or false})$$

H_0 can be tested by use of the test statistic

$$F = \frac{(n-1)\, Q_H}{Q_E} \sim F(d-1,\ (n-1)(d-1)) \quad \text{when } H_0 \text{ is true}$$

where large values of F_{obs} are critical for H_0