3.4

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$, independent

$\underset{\sim}{\bar{x}} = [\bar{x}_1, \dots, \bar{x}_d]^T$, $S^{-1} = \{ s^{jk} \}$

$H_0: \mu_1 = \dots = \mu_d \; (= \mu) \iff \underset{\sim}{\mu} = \mu \underset{\sim}{1}_d$

Theorem 3.3 shows

$$T^2 = \min_{\underset{\sim}{\mu} \in \mathcal{R}(\underset{\sim}{1}_d)} n(\underset{\sim}{\bar{x}} - \mu \underset{\sim}{1}_d)^T S^{-1} (\underset{\sim}{\bar{x}} - \mu \underset{\sim}{1}_d) \sim T^2(d-1, n-1)$$

From page 79 line 13 we have the solution

$$\mu^* = (\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d)^{-1} (\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{\bar{x}}) = \frac{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{\bar{x}}}{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d} ,$$

and from line 16

$$T^2 = n(\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{\bar{x}} - \underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{1}_d \mu^*)$$

$$= n \underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{\bar{x}} - \frac{n(\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{1}_d)^2}{\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d}$$ (identical with the expression in ex. 3.2)

$$\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{\bar{x}} = \sum_j \sum_k \bar{x}_j s^{jk} \bar{x}_k = \sum_j \sum_k s^{jk} \bar{x}_j \bar{x}_k$$

$$\underset{\sim}{\bar{x}}^T S^{-1} \underset{\sim}{1}_d = \sum_j \sum_k \bar{x}_j s^{jk} 1 = \sum_j \sum_k s^{jk} \bar{x}_j = \sum_k \sum_j s^{kj} \bar{x}_k$$

$$= \sum_j \sum_k s^{jk} \bar{x}_k = \frac{1}{2} \sum_j \sum_k s^{jk} (\bar{x}_j + \bar{x}_k)$$

$$\underset{\sim}{1}_d^T S^{-1} \underset{\sim}{1}_d = \sum_j \sum_k 1 s^{jk} 1 = \sum_j \sum_k s^{jk}$$

$$\mu^* = \frac{\frac{1}{2} \sum_j \sum_k s^{jk} (\bar{x}_j + \bar{x}_k)}{\sum_j \sum_k s^{jk}} = \frac{\sum_j \sum_k s^{jk} (\bar{x}_j + \bar{x}_k)}{2 \sum_j \sum_k s^{jk}}$$

$$T^2 = n \sum_j \sum_k s^{jk} \bar{x}_j \bar{x}_k - \frac{n\left(\frac{1}{2} \sum_j \sum_k s^{jk} (\bar{x}_j + \bar{x}_k)\right)^2}{\sum_j \sum_k s^{jk}}$$

$$= n \sum_j \sum_k s^{jk} \bar{x}_j \bar{x}_k - \frac{n\left(\sum_j \sum_k s^{jk} (\bar{x}_j + \bar{x}_k)\right)^2}{4 \sum_j \sum_k s^{jk}}$$

3.5

Consider $\varphi_i = \mu_i - \mu_d \quad , \quad i = 1, \dots, d-1$

$$\sum_{i=1}^{d-1} h_i \varphi_i = \sum_{i=1}^{d-1} h_i(\mu_i - \mu_d) = \sum_{i=1}^{d-1} h_i \mu_i - \mu_d \sum_{i=1}^{d-1} h_i$$

$$= \sum_{i=1}^{d-1} c_i \mu_i + c_d \mu_d = \sum_{i=1}^{d} c_i \mu_i \quad , \quad \text{where}$$

$$c_i = h_i \quad , \quad i = 1, \dots, d-1 \quad , \quad c_d = -\sum_{i=1}^{d-1} h_i$$

$$\sum_{i=1}^{d} c_i = \sum_{i=1}^{d-1} c_i + c_d = \sum_{i=1}^{d-1} h_i - \sum_{i=1}^{d-1} h_i = 0$$

Consider $\sum_{i=1}^{d} c_i \mu_i \quad , \quad \sum_{i=1}^{d} c_i = 0$

$$\sum_{i=1}^{d} c_i \mu_i = \sum_{i=1}^{d-1} c_i \mu_i + c_d \mu_d = \sum_{i=1}^{d-1} c_i \mu_i + \left(-\sum_{i=1}^{d-1} c_i\right) \mu_d$$

$$= \sum_{i=1}^{d-1} c_i(\mu_i - \mu_d) = \sum_{i=1}^{d-1} h_i \varphi_i \quad , \quad \text{where}$$

$$h_i = c_i \quad , \quad \varphi_i = \mu_i - \mu_d \quad , \quad i = 1, \dots, d-1$$

3.6

$$L_2 = \tfrac{1}{2} \operatorname{tr}\left(Q Q_{(2)}^{-1} - I_d\right)^2 \quad , \quad Q_{(2)} = \begin{array}{cc} \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} & \begin{matrix} d_1 \\ d_2 \end{matrix} \end{array}$$

$$Q Q_{(2)}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} Q_{11}^{-1} & 0 \\ 0 & Q_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I_{d_1} & Q_{12} Q_{22}^{-1} \\ Q_{21} Q_{11}^{-1} & I_{d_2} \end{bmatrix}$$

$$(Q Q_{(2)}^{-1} - I_d)^2 = \begin{bmatrix} 0 & Q_{12} Q_{22}^{-1} \\ Q_{21} Q_{11}^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & Q_{12} Q_{22}^{-1} \\ Q_{21} Q_{11}^{-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1} & 0 \\ 0 & Q_{21} Q_{11}^{-1} Q_{12} Q_{22}^{-1} \end{bmatrix}$$

$$L_2 = \tfrac{1}{2}(n-1)\left(\operatorname{tr}(Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1}) + \operatorname{tr}(Q_{21} Q_{11}^{-1} Q_{12} Q_{22}^{-1})\right)$$

$$= (n-1) \operatorname{tr}(Q_{12} Q_{22}^{-1} Q_{21} Q_{11}^{-1})$$

$$= (n-1) \operatorname{tr}(S_{12} S_{22}^{-1} S_{21} S_{11}^{-1})$$

3.7

$$\underset{\sim}{x} \sim N_2(\underset{\sim}{\mu}, \Sigma) \quad , \quad \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad , \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Theorem 2.1 (viii) shows that

$$x_2 | x_1 \sim N(\mu_2 + \rho\sigma_1\sigma_2\sigma_1^{-2}(x_1-\mu_1) \, , \, \sigma_2^2 - \rho\sigma_1\sigma_2\sigma_1^{-2}\rho\sigma_1\sigma_2)$$

$$= N\left(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1) \, , \, \sigma_2^2(1-\rho^2)\right)$$

($\sigma_2^2(1-\rho^2)$ is called the residual variance)

$$x_2|x_1 = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1) + \varepsilon \quad , \quad \varepsilon \sim N(0, \sigma_2^2(1-\rho^2))$$

$$= \alpha + \beta x_1 + \varepsilon \quad , \quad \alpha = \mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1 \quad , \quad \beta = \rho\frac{\sigma_2}{\sigma_1}$$

$\underset{\sim}{x}_i \sim N_2(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \ldots, n$, independent

$x_{i2} | x_{i1} = \alpha + \beta x_{i1} + \varepsilon_i$, $\varepsilon_i \sim N(0, \sigma_2^2(1-\rho^2))$,
$i = 1, \ldots, n$, independent

$H_0 : \beta = 0$, $H_1 : \beta \neq 0$

Test statistic : $t = \dfrac{\hat{\beta}}{s_{\hat{\beta}}} \sim t(n-2)$ when H_0 is true

$$\hat{\alpha} = \bar{x}_{\cdot 2} - \hat{\beta}\bar{x}_{\cdot 1} \quad , \quad \hat{\beta} = \frac{q_{12}}{q_{11}} \quad , \quad s_{\hat{\beta}}^2 = \frac{\frac{1}{n-2}\sum_i e_i^2}{q_{11}}$$

$$\sum_i e_i^2 = \sum_i (x_{i2} - (\hat{\alpha} + \hat{\beta}x_{i1}))^2$$

$$= \sum_i (x_{i2} - \bar{x}_{\cdot 2} - \hat{\beta}(x_{i1} - \bar{x}_{\cdot 1}))^2$$

$$= q_{22} - 2\hat{\beta}q_{12} + \hat{\beta}^2 q_{11} = q_{22} - \hat{\beta}q_{12}$$

$$= \frac{q_{11}q_{22} - q_{12}^2}{q_{11}}$$

3.7

$$s_{\hat\beta}^2 = \frac{1}{n-2} \frac{q_{11} q_{22} - q_{12}^2}{q_{11}^2}$$

$$t = \frac{\frac{q_{12}}{q_{11}}}{\sqrt{\frac{1}{n-2} \frac{q_{11} q_{22} - q_{12}^2}{q_{11}^2}}} = \frac{\frac{q_{12}}{\sqrt{q_{11}}\sqrt{q_{22}}} \sqrt{n-2}}{\sqrt{\frac{q_{11} q_{22} - q_{12}^2}{q_{11} q_{22}}}}$$

$$= \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

$$f(t) = \frac{1}{\sqrt{(n-2)\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(1 + \frac{t^2}{n-2}\right)^{-\frac{n-1}{2}} , \quad -\infty < t < \infty$$

$$\frac{dt}{dr} = \frac{\sqrt{1-r^2}\sqrt{n-2} - r\sqrt{n-2}\,\frac{1}{2\sqrt{1-r^2}}(-2r)}{1-r^2} = \frac{\sqrt{n-2}\,(1-r^2+r^2)}{(1-r^2)\sqrt{1-r^2}}$$

$$= \frac{\sqrt{n-2}}{(1-r^2)^{\frac{3}{2}}}$$

$$g(r) = \frac{1}{\sqrt{(n-2)\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(1 + \frac{(n-2)r^2}{(n-2)(1-r^2)}\right)^{-\frac{n-1}{2}} \left|\frac{\sqrt{n-2}}{(1-r^2)^{\frac{3}{2}}}\right|$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(\frac{1-r^2+r^2}{1-r^2}\right)^{-\frac{n-1}{2}} \left(\frac{1}{1-r^2}\right)^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} (1-r^2)^{\frac{n-4}{2}} , \quad -1 \le r \le 1$$