Test for blockwise independence / b blocks

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$, independent , $n-1 \geq d$

Partition :

$$\underset{\sim}{x}_i = \begin{bmatrix} \underset{\sim}{x}_{i1} \\ \underset{\sim}{x}_{i2} \\ \vdots \\ \underset{\sim}{x}_{ib} \end{bmatrix}, \quad \underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \\ \vdots \\ \underset{\sim}{\mu}_b \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1b} \\ \Sigma_{21} & \Sigma_{22} & \dots & \Sigma_{2b} \\ \vdots & \vdots & & \vdots \\ \Sigma_{b1} & \Sigma_{b2} & \dots & \Sigma_{bb} \end{bmatrix}$$

$\underset{\sim}{x}_{ir}$ d_r-dim , Σ_{rr} $d_r \times d_r$, $r = 1, \dots, b$, $\sum_r d_r = d$

H_0 : $\underset{\sim}{x}_{i1}, \underset{\sim}{x}_{i2}, \dots, \underset{\sim}{x}_{ib}$ independent $\Leftrightarrow$ $\Sigma_{ir} = 0$, $i \neq r$
cf th. 2.1 (iv)

By calculations analogue to the case of two blocks we find

$$\Lambda = \ell^{\frac{2}{n}} = \frac{\det Q}{\det Q_{11} \det Q_{22} \dots \det Q_{bb}} ,$$

but the distribution of Λ can not be specified exactly

No. of parameters in $\underset{\sim}{\mu}$ and Σ : $d + \left(\frac{d^2-d}{2} + d\right) = d + \frac{d^2+d}{2}$

$= d + \frac{1}{2}d(d+1) = \frac{1}{2}d(d+3)$

— — , H_0 true : $\frac{1}{2}\sum_{r=1}^{b} d_r(d_r+3)$

$= \frac{1}{2}\sum_r d_r^2 + \frac{3}{2}d$

$-2\ln \ell \sim \chi^2(\nu)$ approximately when H_0 is true

$$\nu = \frac{1}{2}d(d+3) - \frac{1}{2}\left(\sum_r d_r^2 + 3d\right)$$
$$= \frac{1}{2}\left(d^2 - \sum_r d_r^2\right)$$

Thus $-n \ln \Lambda \sim \chi^2\left(\frac{1}{2}\left(d^2 - \sum_r d_r^2\right)\right)$ appr. when H_0 is true

A better approximation can be found in the book p. 91

Nagaos test statistic

$L_b = \frac{1}{2}(n-1)\,\mathrm{tr}\left(Q\,Q_{(b)}^{-1} - I_d\right)^2$, where

$$Q_{(b)} = \ulcorner Q_{11}\ Q_{22}\ \cdots\ Q_{bb} \lrcorner$$

Test for diagonal variance matrix

H_{01} : $\Sigma = \ulcorner \sigma_1^2\ \sigma_2^2 \cdots \sigma_d^2 \lrcorner$ (blockwise indep., $d_r = 1$)

Let $b = d$, $\underset{\frown}{x}_{ir} = x_{ir}$, $Q_{rr} = q_{rr} = \sum_i (x_{ir} - \bar{x}_{\cdot r})^2$

$\Lambda_1 = \dfrac{\det Q}{q_{11}\, q_{22} \cdots q_{dd}}$, $-n \ln \Lambda_1 \sim \chi^2\left(\frac{1}{2} d(d-1)\right)$ appr. when H_{01} is true

Emperical correlation coefficient between x_{ij} and x_{ik}

$r_{jk} = \dfrac{q_{jk}}{\sqrt{q_{jj}}\sqrt{q_{kk}}}$, choose $R = \{r_{jk}\}$ sym., i.e.

$$R = \ulcorner \frac{1}{\sqrt{q_{11}}} \cdots \frac{1}{\sqrt{q_{dd}}} \lrcorner\ Q\ \ulcorner \frac{1}{\sqrt{q_{11}}} \cdots \frac{1}{\sqrt{q_{dd}}} \lrcorner$$

$\Lambda_1 = \det R$

A special case :

H_{02} : $\Sigma = \sigma^2 I_d$, σ^2 unknown parameter

$\Lambda_2 = \dfrac{\det Q}{\left(\mathrm{tr}\,\frac{Q}{d}\right)^d}$ cf. exercise 3.8

H_{02} is equivalent to a test for all eigenvalues of Σ equal

$$\lambda_1 = \ldots = \lambda_d \Leftrightarrow \left(\prod_j \lambda_j\right)^{\frac{1}{d}} = \frac{1}{d}\sum_j \lambda_j \Leftrightarrow \frac{(\det \Sigma)^{\frac{1}{d}}}{\frac{1}{d}\operatorname{tr}\Sigma} = 1$$

Σ cf. p. 6

Determine a test statistic by exchanging Σ with $\hat{\Sigma}$, hence

reject when $\dfrac{(\det \hat{\Sigma})^{\frac{1}{d}}}{\operatorname{tr}\frac{\hat{\Sigma}}{d}} = \dfrac{(\det Q)^{\frac{1}{d}}}{\operatorname{tr}\frac{Q}{d}} = \Lambda_2^{\frac{1}{d}} \ll 1$

H_{02} is called the spherical hypothesis, as the ellipse $(\underset{\sim}{x}-\underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}) = c^2$ is reduced to $(\underset{\sim}{x}-\underset{\sim}{\mu})^T (\underset{\sim}{x}-\underset{\sim}{\mu}) = \sigma^2 c^2$ when H_{02} is true

H_{03}: $\Sigma = I_d$

$$l_3 = \left(\frac{e}{n}\right)^{\frac{nd}{2}} (\det Q)^{\frac{n}{2}} \exp\left(-\tfrac{1}{2}\operatorname{tr} Q\right) \quad \text{cf. ex. 3.9}$$

When H_{03} is true $-2\ln l_3 \sim \chi^2(\nu)$ approximately,

$$\nu = \tfrac{1}{2}d(d+3) - d = \tfrac{1}{2}d(d+1)$$

H_{02}': $\Sigma = \sigma^2 \Sigma_0$ and H_{03}': $\Sigma = \Sigma_0$, Σ_0 known, is reduced to H_{02} and H_{03} respectively by the transformation $\underset{\sim}{y_i} = \Sigma_0^{-\frac{1}{2}} \underset{\sim}{x_i}$, as

$$\operatorname{Var} \underset{\sim}{y_i} = \Sigma_0^{-\frac{1}{2}} \sigma^2 \Sigma_0 \Sigma_0^{-\frac{1}{2}} = \sigma^2 I_d$$

H_{04}: $\underset{\sim}{\mu} = \underset{\sim}{\mu_0}$, $\Sigma = \Sigma_0$ is refered to ex. 3.10

Test for equal diagonal blocks under the assumption that all covariance blocks are equal

Σ is partitioned in b^2 $d_0 \times d_0$ blocks , $b d_0 = d$

Given : $\Sigma_{rs} = \Sigma_1$, $r \neq s$ ($\Rightarrow \Sigma_1$ symmetric)

H_{0S} : $\Sigma_{rr} = \Sigma_0$, $r = 1, \ldots, b$

Let $\bar{\underset{\sim}{x}}_{i\cdot} = \frac{1}{b}(\underset{\sim}{x}_{i1} + \underset{\sim}{x}_{i2} + \ldots + \underset{\sim}{x}_{ib})$

and $\underset{\sim}{y}_i = \begin{bmatrix} \underset{\sim}{y}_{i1} \\ \underset{\sim}{y}_{i2} \\ \vdots \\ \underset{\sim}{y}_{ib} \end{bmatrix} = \begin{bmatrix} \bar{\underset{\sim}{x}}_{i\cdot} \\ \underset{\sim}{x}_{i2} - \bar{\underset{\sim}{x}}_{i\cdot} \\ \vdots \\ \underset{\sim}{x}_{ib} - \bar{\underset{\sim}{x}}_{i\cdot} \end{bmatrix} = C \underset{\sim}{x}_i$

C is non-singular , cf. ex. 3.11

$\underset{\sim}{y}_i \sim N_d(C\underset{\sim}{\mu}, C \Sigma C^T)$ cf. theorem 2.1 (i)

$$\begin{aligned}
\mathrm{Cov}(\underset{\sim}{y}_{ir}, \underset{\sim}{y}_{i1}) &= \mathrm{Cov}(\underset{\sim}{x}_{ir} - \bar{\underset{\sim}{x}}_{i\cdot}, \bar{\underset{\sim}{x}}_{i\cdot}) \qquad , \quad r = 2, \ldots, b \\
&= \mathrm{Cov}(\underset{\sim}{x}_{ir}, \bar{\underset{\sim}{x}}_{i\cdot}) - \mathrm{Var}\, \bar{\underset{\sim}{x}}_{i\cdot} \\
&= \frac{1}{b} \sum_s \mathrm{Cov}(\underset{\sim}{x}_{ir}, \underset{\sim}{x}_{is}) - \frac{1}{b^2} \sum_q \sum_s \mathrm{Cov}(\underset{\sim}{x}_{iq}, \underset{\sim}{x}_{is}) \\
&= \frac{1}{b}\left((b-1)\Sigma_1 + \Sigma_0\right) - \frac{1}{b^2}\left(b(b-1)\Sigma_1 + b\Sigma_0\right) \\
&= 0 \qquad (d_0 \times d_0)
\end{aligned}$$

Let $\underset{\sim}{y}_{i2}^* = \begin{bmatrix} \underset{\sim}{y}_{i2} \\ \vdots \\ \underset{\sim}{y}_{ib} \end{bmatrix} \Rightarrow \underset{\sim}{y}_i = \begin{bmatrix} \underset{\sim}{y}_{i1} \\ \underset{\sim}{y}_{i2}^* \end{bmatrix} \begin{array}{l} d_0\text{-dim.} \\ (b-1)d_0\text{-dim.} \end{array}$

$$\text{Cov}(\underline{y}_{i1}, \underline{y}_{i2}^*) = 0 \qquad ((b-1)d_0 \times d_0)$$

H_{05} is equivalent with testing for independence between $\underline{y}_{i1}$ og $\underline{y}_{i2}^*$ (blockwise indep. / 2 blocks)

In the case $b = 2$ the presumption becomes
$\Sigma_{12} = \Sigma_{21} \; (= \Sigma_1)$

Test for equal variances and equal correlations

$$H_{06}: \quad \Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & & 1 \end{bmatrix}$$

Maximum likelihood estimates:

$$\left.\begin{aligned} \hat{\sigma}^2 &= \frac{1}{d} \sum_j s^*_{jj} \\ \widehat{\sigma^2 \rho} &= \frac{1}{d(d-1)} \sum_j \sum_{k \neq j} s^*_{jk} \end{aligned}\right\} \quad \text{where } S^* = \frac{Q}{n},$$

cf. exercise 3.12

The likelihood ratio test statistic

$$l_6^{\frac{2}{n}} = \frac{\det S^*}{(\hat{\sigma}^2)^d (1-\hat{\rho})^{d-1} (1+(d-1)\hat{\rho})} \qquad \text{cf. ex. 3.13}$$

$$-k \ln l_6 \sim \chi^2\left(\tfrac{1}{2}d(d+1) - 2\right) \quad \underline{\text{app.}} \text{ when } H_0 \text{ is true}$$

cf. page 95 bottom

Establishing $\left(\prod_j \lambda_j\right)^{\frac{1}{d}} \leq \frac{1}{d}\sum_j \lambda_j$:

Jensen's inequality: $\varphi\left(\int_\Omega f\,d\mu\right) \leq \int_\Omega (\varphi\circ f)\,d\mu$, φ convex

The stochastic equivalent: $\varphi(EX) \leq E[\varphi\circ X]$, φ convex

Consider a discrete stoch. var. X uniformly distributed on $\{x_1, \dots, x_d\}$

i.e. $P(X = x_j) = \frac{1}{d}$, $j = 1, \dots, d$.

$$EX = \sum_j x_j \frac{1}{d} = \frac{1}{d}\sum_j x_j$$

Applied to the convex function exp we get

$$E[\exp X] = \sum_j e^{x_j}\frac{1}{d} = \frac{1}{d}\sum_j e^{x_j}$$

$$\exp(EX) = e^{\frac{1}{d}\sum_j x_j} = \left(\prod_j e^{x_j}\right)^{\frac{1}{d}}$$

Jensen's inequality: $\left(\prod_j e^{x_j}\right)^{\frac{1}{d}} \leq \frac{1}{d}\sum_j e^{x_j}$

With $e^{x_j} = \lambda_j$: $\left(\prod_j \lambda_j\right)^{\frac{1}{d}} \leq \frac{1}{d}\sum_j \lambda_j$

Note that the equality sign is valid when $\lambda_1 = \dots = \lambda_d \ (= \lambda)$

Two sets of observations

Test of equal variance matrix

$$\underset{\sim}{v}_i \sim N_d(\underset{\sim}{\mu}_1, \Sigma_1), \quad i = 1, \dots, n_1$$
$$\underset{\sim}{w}_j \sim N_d(\underset{\sim}{\mu}_2, \Sigma_2), \quad j = 1, \dots, n_2$$
all independent

$$L(\hat{\underset{\sim}{\mu}}_1, \hat{\underset{\sim}{\mu}}_2, \hat{\Sigma}_1, \hat{\Sigma}_2) = L_1(\hat{\underset{\sim}{\mu}}_1, \hat{\Sigma}_1)\, L_2(\hat{\underset{\sim}{\mu}}_2, \hat{\Sigma}_2), \quad \hat{\underset{\sim}{\mu}}_1 = \bar{\underset{\sim}{v}}, \quad \hat{\underset{\sim}{\mu}}_2 = \bar{\underset{\sim}{w}}$$
$$= (2\pi e)^{-\frac{n_1 d}{2}} (\det \hat{\Sigma}_1)^{-\frac{n_1}{2}} (2\pi e)^{-\frac{n_2 d}{2}} (\det \hat{\Sigma}_2)^{-\frac{n_2}{2}}$$
$$= (2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma}_1)^{-\frac{n_1}{2}} (\det \hat{\Sigma}_2)^{-\frac{n_2}{2}}, \quad n = n_1 + n_2$$

$H_0: \Sigma_1 = \Sigma_2 \; (= \Sigma)$, $\quad H_1: \Sigma_1 \neq \Sigma_2$

When H_0 is true:

$$\ln L(\hat{\underset{\sim}{\mu}}_1, \hat{\underset{\sim}{\mu}}_2, \Sigma) = \ln L_1(\hat{\underset{\sim}{\mu}}_1, \Sigma) + \ln L_2(\hat{\underset{\sim}{\mu}}_2, \Sigma), \quad \hat{\underset{\sim}{\mu}}_1 = \bar{\underset{\sim}{v}}, \quad \hat{\underset{\sim}{\mu}}_2 = \bar{\underset{\sim}{w}}$$
$$= -\frac{n_1 d}{2}\ln(2\pi) - \frac{1}{2}\left(n_1 \ln(\det \Sigma) + \operatorname{tr}(\Sigma^{-1} Q_1)\right)$$
$$- \frac{n_2 d}{2}\ln(2\pi) - \frac{1}{2}\left(n_2 \ln(\det \Sigma) + \operatorname{tr}(\Sigma^{-1} Q_2)\right)$$
$$= -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\left(n \ln(\det \Sigma) + \operatorname{tr}(\Sigma^{-1} Q)\right),$$
$$Q = Q_1 + Q_2$$

$\hat{\Sigma} = \frac{Q}{n}$ cf. formula (3.5) page 61

$$L(\hat{\underset{\sim}{\mu}}_1, \hat{\underset{\sim}{\mu}}_2, \hat{\Sigma}) = (2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma})^{-\frac{n}{2}}$$

$$\ell = \frac{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma})^{-\frac{n}{2}}}{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma}_1)^{-\frac{n_1}{2}} (\det \hat{\Sigma}_2)^{-\frac{n_2}{2}}} = \frac{(\det \hat{\Sigma}_1)^{\frac{n_1}{2}} (\det \hat{\Sigma}_2)^{\frac{n_2}{2}}}{(\det \hat{\Sigma})^{\frac{n}{2}}}$$
$$= \frac{n^{\frac{nd}{2}}}{n_1^{\frac{n_1 d}{2}}\, n_2^{\frac{n_2 d}{2}}} \; \frac{(\det Q_1)^{\frac{n_1}{2}} (\det Q_2)^{\frac{n_2}{2}}}{(\det(Q_1 + Q_2))^{\frac{n}{2}}}$$

$-2\ln \ell \sim \chi^2(\nu)$ appr., $\quad \nu = 2\,\frac{1}{2}d(d+3) - \left(\frac{1}{2}d(d+3) + d\right)$
$$= \frac{1}{2}d(d+1)$$

$$\forall \underset{\sim}{l}: \quad \left.\begin{array}{l} \underset{\sim}{l}^T Q_1 \underset{\sim}{l} \sim \underset{\sim}{l}^T \Sigma_1 \underset{\sim}{l} \ \chi^2(n_1-1) \\ \underset{\sim}{l}^T Q_2 \underset{\sim}{l} \sim \underset{\sim}{l}^T \Sigma_2 \underset{\sim}{l} \ \chi^2(n_2-1) \end{array}\right\} \text{ independent}$$

$$H_{0l}: \ \underset{\sim}{l}^T \Sigma_1 \underset{\sim}{l} = \underset{\sim}{l}^T \Sigma_2 \underset{\sim}{l}, \qquad \bigcap_l H_{0l} = H_0$$

$$H_{0l} \text{ true}: \quad \frac{\dfrac{\underset{\sim}{l}^T Q_1 \underset{\sim}{l}}{(n_1-1)\,\underset{\sim}{l}^T \Sigma_1 \underset{\sim}{l}}}{\dfrac{\underset{\sim}{l}^T Q_2 \underset{\sim}{l}}{(n_2-1)\,\underset{\sim}{l}^T \Sigma_2 \underset{\sim}{l}}} = \frac{\underset{\sim}{l}^T S_1 \underset{\sim}{l}}{\underset{\sim}{l}^T S_2 \underset{\sim}{l}} \sim F(n_1-1, n_2-1)$$

Acceptance area for H_0:

$$\bigcap_l \left\{ (V,W) : F_{\alpha_1} < \frac{\underset{\sim}{l}^T S_1 \underset{\sim}{l}}{\underset{\sim}{l}^T S_2 \underset{\sim}{l}} < F_{1-\alpha_2} \right\}, \quad \alpha_1 + \alpha_2 = \alpha$$

$$= \bigcap_l \left\{ (V,W) : \frac{n_1-1}{n_2-1} F_{\alpha_1} < \frac{\underset{\sim}{l}^T Q_1 \underset{\sim}{l}}{\underset{\sim}{l}^T Q_2 \underset{\sim}{l}} < \frac{n_1-1}{n_2-1} F_{1-\alpha_2} \right\}$$

$$= \left\{ (V,W) : \frac{n_1-1}{n_2-1} F_{\alpha_1} \le \varphi_{min} \le \varphi_{max} \le \frac{n_1-1}{n_2-1} F_{1-\alpha_2} \right\},$$

where φ is eigenvalues for $Q_1 Q_2^{-1}$, cf. A 7.5

$$= \left\{ (V,W) : c_1 \le \Theta_{min} \le \Theta_{max} \le c_2 \right\},$$

where Θ is eigenvalues for $Q_1(Q_1+Q_2)^{-1}$, $\Theta = \frac{\varphi}{1+\varphi}$

Critical area for H_0:

$$\left\{ (V,W) : \Theta_{min} < c_1 \ \vee \ \Theta_{max} > c_2 \right\}$$

The choice of c_1 and c_2 which gives maximum test strength is not known