Test for equal means assuming equal variance matrices

$$\left.\begin{array}{ll} \underset{\sim}{v}_i \sim N_d(\underset{\sim}{\mu}_1, \Sigma), & i = 1, \dots, n_1 \\ \underset{\sim}{w}_j \sim N_d(\underset{\sim}{\mu}_2, \Sigma), & j = 1, \dots, n_2 \end{array}\right\} \text{ all independent}$$

$$\left.\begin{array}{l} \bar{\underset{\sim}{v}} \sim N_d\left(\underset{\sim}{\mu}_1, \frac{\Sigma}{n_1}\right) \\ \bar{\underset{\sim}{w}} \sim N_d\left(\underset{\sim}{\mu}_2, \frac{\Sigma}{n_2}\right) \\ Q_1 = (n_1 - 1) S_1 \sim W_d(n_1 - 1, \Sigma) \\ Q_2 = (n_2 - 1) S_2 \sim W_d(n_2 - 1, \Sigma) \end{array}\right\} \text{ all independent}$$

$$\left.\begin{array}{l} \bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} \sim N_d\left(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma\right) \\ Q = Q_1 + Q_2 \sim W_d(n_1 + n_2 - 2, \Sigma) \end{array}\right\} \text{ independent}$$

$H_0: \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 \iff \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2 = \underset{\sim}{0}$, $H_1: \underset{\sim}{\mu}_1 \neq \underset{\sim}{\mu}_2$

Let $S_p = \dfrac{Q}{n_1 + n_2 - 2} = \dfrac{Q_1 + Q_2}{n_1 + n_2 - 2} = \dfrac{(n_1 - 1) S_1 + (n_2 - 1) S_2}{n_1 + n_2 - 2}$

When H_0 is true the corollary to theorem 2.8 shows

$$T^2 = \frac{1}{\frac{1}{n_1} + \frac{1}{n_2}} (n_1 + n_2 - 2)(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} - \underset{\sim}{0})^T Q^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} - \underset{\sim}{0})$$

$$= \frac{n_1 n_2}{n_1 + n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T S_p^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}}) \sim T^2(d, n_1 + n_2 - 2)$$

$$= \frac{(n_1 + n_2 - 2) d}{n_1 + n_2 - d - 1} F(d, n_1 + n_2 - d - 1)$$

Let $n_1 + n_2 = n$ and notice for $n \to \infty$:

$$T^2(d, n-2) = \frac{(n-2) d}{n - d - 1} F(d, n - d - 1) \to d\, F(d, \infty)$$

$$= \chi^2(d)$$

i.e. $T^2 \sim \chi^2(d)$ app. when n is large and H_0 is true

A 7.6 shows

$$T^2 = \frac{n_1 n_2}{n_1+n_2} \sup_{\underset{\sim}{\ell}} \frac{\left(\underset{\sim}{\ell}^T(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})\right)^2}{\underset{\sim}{\ell}^T S_p \underset{\sim}{\ell}} = \sup_{\underset{\sim}{\ell}} \left(\frac{\underset{\sim}{\ell}^T(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})}{\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})\underset{\sim}{\ell}^T S_p \underset{\sim}{\ell}}} \right)^2$$

$$= \sup_{\underset{\sim}{\ell}} t_\ell^2$$

$t_\ell \sim t(n_1+n_2-2)$ is the usual test statistic for test of equal means in two normal distributed sets of observations assuming equal variances,

here
$$\left.\begin{array}{ll} \underset{\sim}{\ell}^T \underset{\sim}{v}_i \sim N(\underset{\sim}{\ell}^T \underset{\sim}{\mu}_1, \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}) & , i = 1, \dots, n_1 \\ \underset{\sim}{\ell}^T \underset{\sim}{w}_j \sim N(\underset{\sim}{\ell}^T \underset{\sim}{\mu}_2, \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}) & , j = 1, \dots, n_2 \end{array}\right\} \text{ all indep.}$$

and $H_{0\ell}: \underset{\sim}{\ell}^T \underset{\sim}{\mu}_1 = \underset{\sim}{\ell}^T \underset{\sim}{\mu}_2 \Leftrightarrow \underset{\sim}{\ell}^T(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) = 0$

$H_0 = \bigcap_\ell H_{0\ell}$ which shows that the $\cup$-$\cap$-principal would produce the same test statistic

Confidence intervals

$$P\left(\frac{n_1 n_2}{n_1+n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} - (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2))^T S_p^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} - (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)) \le T^2_{1-\alpha}(d, n_1+n_2-2) \right) = 1-\alpha$$

can be used for construction of confidence intervals, e.g.

$$\underset{\sim}{h}^T(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2) = \underset{\sim}{h}^T(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}}) \pm \sqrt{T^2_{1-\alpha} \left(\frac{1}{n_1}+\frac{1}{n_2}\right) \underset{\sim}{h}^T S_p \underset{\sim}{h}}$$

with confidence level $1-\alpha$ for all $h^T(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)$ (Scheffé - intervals)

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Ex.

EXAMPLE 3.6 Using individual measurements of cranial length (x_1) and breadth (x_2) on 35 mature female frogs (*Rana esculenta*) and 14 mature male frogs published in a study by Kauri, Reyment [1961] obtained the following results:

$$\bar{\mathbf{v}} = \begin{pmatrix} 22.860 \\ 24.397 \end{pmatrix}, \qquad \bar{\mathbf{w}} = \begin{pmatrix} 21.821 \\ 22.843 \end{pmatrix}$$

$$\mathbf{S}_1 = \begin{pmatrix} 17.683 & 20.292 \\ 20.292 & 24.407 \end{pmatrix}, \qquad \mathbf{S}_2 = \begin{pmatrix} 18.479 & 19.095 \\ 19.095 & 20.755 \end{pmatrix}$$

and

$$\mathbf{S}_p = \frac{(n_1 - 1)\mathbf{S}_1 + (n_2 - 1)\mathbf{S}_2}{n_1 + n_2 - 2} = \begin{pmatrix} 17.903 & 19.959 \\ 19.959 & 23.397 \end{pmatrix}.$$

We note that $\mathbf{S}_1$ and $\mathbf{S}_2$ are very similar so that the assumption of $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ is tenable. To test the difference between the population means for female and male frogs we can use (3.86) with $d = 2$, $n_1 = 35$, and $n_2 = 14$, that is, $T_0^2 = 1.9693$. From (3.87) we therefore compare

$$F_0 = \frac{n_1 + n_2 - d - 1}{d(n_1 + n_2 - 2)} T_0^2 = \frac{46}{2(47)} T_0^2 = 0.964$$

with $F_{2,46}^{0.05} = 3.2$ and conclude that the test is not significant.

Although the null hypothesis of equal means is not rejected, it is instructive to demonstrate the calculation of confidence intervals. Writing $\boldsymbol{\mu}_1' = (\mu_{F_1}, \mu_{F_2})$ and $\boldsymbol{\mu}_2' = (\mu_{M_1}, \mu_{M_2})$, we can use (3.90) with

$$T^2_{d, n_1 + n_2 - 2, .05} = \frac{2(47)}{46} F_{2,46}^{0.05} = 6.54$$

for constructing the confidence interval for any linear combination of the form $h_1(\mu_{F1} - \mu_{M1}) + h_2(\mu_{F2} - \mu_{M2})$. In particular, setting $\mathbf{h}'$ equal to $(1, 0)$, we have $\mathbf{h}'\mathbf{S}_p\mathbf{h} = s_{11}$, the first diagonal element of $\mathbf{S}_p$, and a confidence interval for $\mu_{F1} - \mu_{M1}$ is

$$22.860 - 21.821 \pm 6.54^{1/2}\left[\left(\tfrac{1}{35} + \tfrac{1}{14}\right)17.903\right]^{1/2}$$

or 1.04 ± 3.42. Similarly, setting $\mathbf{h}' = (0, 1)$, we can find a confidence interval for $\mu_{F_2} - \mu_{M_2}$, namely, 1.55 ± 3.91. The two intervals have a combined "confidence" of at least 95%.

If we had decided before seeing the data that we required these two confidence intervals, then we could use the Bonferroni method with $t_{47}^{0.05/4} =$ 1)
2.31 (see Appendix D1) in the above intervals instead of $(T^2_{2,47,0.05})^{1/2} = 6.54^{1/2} = 2.56$. For example, the first interval now becomes 1.04 ± 3.09.

In conclusion we note, from $\mathbf{S}_p$, that an estimate of the correlation between cranial length and breadth is $19.959(17.903 \times 23.397)^{-1/2} = 0.98$, thus suggesting that only one measurement is sufficient. 2)

1) $t_{1-\frac{\alpha}{2k}}(\nu) = t_{1-\frac{\alpha}{2\cdot 2}}(47)$ $\quad (k = 2)$

2) $r = \dfrac{s_{12}}{\sqrt{s_{11}\, s_{22}}} = \dfrac{19{,}959}{\sqrt{17{,}903 \cdot 23{,}397}}$

When $\Sigma_1 \neq \Sigma_2$?

Assume that n_1 and n_2 are reasonable large hence $S_1 \approx \Sigma_1$ and $S_2 \approx \Sigma_2$

$$T^2 = \frac{n_1 n_2}{n_1 + n_2} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T \left(\frac{(n_1-1) S_1 + (n_2-1) S_2}{n_1+n_2-2} \right)^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})$$

$$= \frac{n_1+n_2-2}{n_1+n_2} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T \left(\frac{(n_1-1) S_1 + (n_2-1) S_2}{n_1 n_2} \right)^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})$$

$$\approx (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T \left(\frac{\Sigma_1}{n_2} + \frac{\Sigma_2}{n_1} \right)^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}}) = D \quad \text{(short notation)}$$

From ealier : $\bar{v} - \bar{w} \sim N_d \left(\underset{\sim}{\mu_1} - \underset{\sim}{\mu_2}, \frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2} \right)$

$$\Rightarrow \quad y = (\bar{v} - \bar{w})^T \left(\frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2} \right)^{-1} (\bar{v} - \bar{w})$$

$$\sim \chi^2 \left(d ;\ (\underset{\sim}{\mu_1} - \underset{\sim}{\mu_2})^T \left(\frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2} \right)^{-1} (\underset{\sim}{\mu_1} - \underset{\sim}{\mu_2}) \right) \quad \text{cf. th. 2.1 (viii)}$$

$y \sim \chi^2 (d)$ when H_0 is true

For $\underline{n_1 = n_2}$ we see $D = y$, whether $\Sigma_1 = \Sigma_2$ or not. Thus when $n_1 = n_2$ T^2 is not influenced seriously by deviation from $\Sigma_1 = \Sigma_2$.

Note that for $n_1 = n_2$ we have $S_p = \frac{S_1 + S_2}{2}$, hence

$$T^2 = \frac{n^2}{2n} (\bar{v} - \bar{w})^T \, 2 \, (S_1 + S_2)^{-1} (\bar{v} - \bar{w}) , \quad n = n_1 = n_2$$

$$= n (\bar{v} - \bar{w})^T (S_1 + S_2)^{-1} (\bar{v} - \bar{w}) \sim T^2 (d, 2(n-1))$$

Now assume $n_1 \neq n_2$, n_1 and n_2 large

According to James

$(\bar{v} - \bar{w})^T \left(\frac{S_1}{n_1} + \frac{S_2}{n_2} \right)^{-1} (\bar{v} - \bar{w}) = (\bar{v} - \bar{w}) S_T^{-1} (\bar{v} - \bar{w}) \sim \chi^2(d)$ $\underline{\text{app.}}$

can be used as test statistic

This is a generalization of Welsh's test for equal means in two one-dimensional normal distributed sets of observations with different variances $\left(\frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim t(f) \text{ app.} \right)$

Approximative critical value (James)

$$k_{1-\alpha} = \chi^2_{1-\alpha}(d) \left(A + B\, \chi^2_{1-\alpha}(d) \right), \text{ where}$$

$$A = 1 + \frac{1}{2d} \sum_{i=1}^{2} \frac{1}{n_i - 1} \left(\mathrm{tr}\left(\frac{S_T^{-1} S_i}{n_i} \right) \right)^2$$

$$B = \frac{1}{d(d+2)} \left(\sum_{i=1}^{2} \frac{1}{n_i - 1} \left(\frac{1}{2} \left(\mathrm{tr}\left(\frac{S_T^{-1} S_i}{n_i} \right) \right)^2 - \mathrm{tr}\left(\frac{S_T^{-1} S_i}{n_i} \right)^2 \right) \right)$$

$$S_T = \frac{S_1}{n_1} + \frac{S_2}{n_2}$$

Alternatively (Yao)

$$(\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T \left(\frac{S_1}{n_1} + \frac{S_2}{n_2} \right)^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}}) \sim T^2(d, f) \text{ app.},$$

$$\text{where } \frac{1}{f} = \sum_{i=1}^{2} \frac{1}{n_i - 1} \left(\frac{(\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T S_T^{-1} \frac{S_i}{n_i} S_T^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})}{(\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T S_T^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})} \right)^2 \text{ app.}$$

cf. the Welsh test for one dimensional sets of observations, where $\frac{1}{f} = \sum_{i=1}^{2} \frac{1}{n_i - 1} \left(\frac{\frac{s_i^2}{n_i}}{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)^2$ app.

EXAMPLE 3.7 From James [1954] we have the following data:

$$\bar{\mathbf{v}} = \begin{pmatrix} 9.82 \\ 15.06 \end{pmatrix}, \quad \bar{\mathbf{w}} = \begin{pmatrix} 13.05 \\ 22.57 \end{pmatrix}, \quad n_1 = 16, n_2 = 11,$$

$$\mathbf{S}_1 = \begin{pmatrix} 120.0 & -16.3 \\ -16.3 & 17.8 \end{pmatrix} \text{ and } \mathbf{S}_2 = \begin{pmatrix} 81.8 & 32.1 \\ 32.1 & 53.8 \end{pmatrix}.$$

From (3.98),

$$\mathbf{S}_T = \begin{pmatrix} 14.936 & 1.899 \\ 1.899 & 6.003 \end{pmatrix}$$

and

$$(\bar{\mathbf{v}} - \bar{\mathbf{w}})'\mathbf{S}_T^{-1}(\bar{\mathbf{v}} - \bar{\mathbf{w}}) = (-3.23, -7.51)\begin{pmatrix} 0.06976 & -0.02207 \\ -0.02207 & 0.17357 \end{pmatrix}\begin{pmatrix} -3.23 \\ -7.51 \end{pmatrix}$$

$$= 9.45.$$

Also

$$\mathbf{B}_1 = \frac{1}{n_1}\mathbf{S}_T^{-1}\mathbf{S}_1 = \begin{pmatrix} 0.5457 & -0.0956 \\ -0.3424 & 0.2155 \end{pmatrix},$$

$$\mathbf{B}_2 = \frac{1}{n_2}\mathbf{S}_T^{-1}\mathbf{S}_2 = \begin{pmatrix} 0.4543 & 0.0956 \\ 0.3424 & 0.7845 \end{pmatrix};$$

$$\operatorname{tr}\mathbf{B}_1^2 = 0.4097, \quad (\operatorname{tr}\mathbf{B}_1)^2 = 0.5794, \quad \operatorname{tr}\mathbf{B}_2^2 = 0.8873, \quad (\operatorname{tr}\mathbf{B}_2)^2 = 1.5346;$$

$$\sum_i \frac{1}{n_i - 1}\operatorname{tr}\mathbf{B}_i^2 = 0.11604, \qquad \sum_i \frac{1}{n_i - 1}(\operatorname{tr}\mathbf{B}_i)^2 = 0.19209;$$

$A = 1 + \frac{1}{4}(0.19209) = 1.0480$ and $B = \frac{1}{8}\left[\frac{1}{2}(0.19209) + 0.11604\right] = 0.02651.$

Values of k_α are given in Table 3.6.

For the approximate degrees of freedom method we have (Yao [1965])

$$\frac{1}{f} = \frac{1}{15}(0.1657)^2 + \frac{1}{10}(0.8343)^2$$

TABLE 3.6 Critical Values for the James (k_α) and Yao (T_α^2) Tests

α	$\chi_2^2(\alpha)$	k_α	T_α^2
0.05	5.991	7.23	8.21
0.025	7.378	9.18	10.70
0.01	9.210	11.90	14.43

and $f = 14.00$. From (2.47)

$$T_{d,f,\alpha}^2 = \frac{df}{f - d + 1}F_{d,f-d+1}^\alpha \qquad (= T_\alpha^2, \text{ say})$$

and this is also tabled in Table 3.6. As expected, the critical values of the two tests are very similar and, in general, the degrees of freedom method is preferred, as it is more conservative. For either test, the value of the test statistic, 9.45, is significant at 5%. When f is not an integer, we can use the tables of Mardia and Zemroch [1975b].

Profile analysis / two populations

Consider a population with mean vector $\underset{\sim}{\eta} = [\eta_1\ \eta_2 \ldots \eta_d]^T$

The graph obtained by successively joining the points $(1, \eta_1), (2, \eta_2), \ldots, (d, \eta_d)$ by straight lines is called the profile of the population.

Samples from two populations with equal variance matrix:

$$\left.\begin{array}{ll} \underset{\sim}{v}_i \sim N_d(\underset{\sim}{\eta}, \Sigma), & i = 1, \ldots, n_1 \\ \underset{\sim}{w}_j \sim N_d(\underset{\sim}{\nu}, \Sigma), & j = 1, \ldots, n_2 \end{array}\right\} \text{ all independent}$$

H_{01}: The two profiles are parallel

$\Leftrightarrow \eta_k - \eta_{k-1} = \nu_k - \nu_{k-1}, \quad k = 2, \ldots, d$

$\Leftrightarrow C_1 \underset{\sim}{\eta} = C_1 \underset{\sim}{\nu}$ (C_1 as shown in formula (3.18))

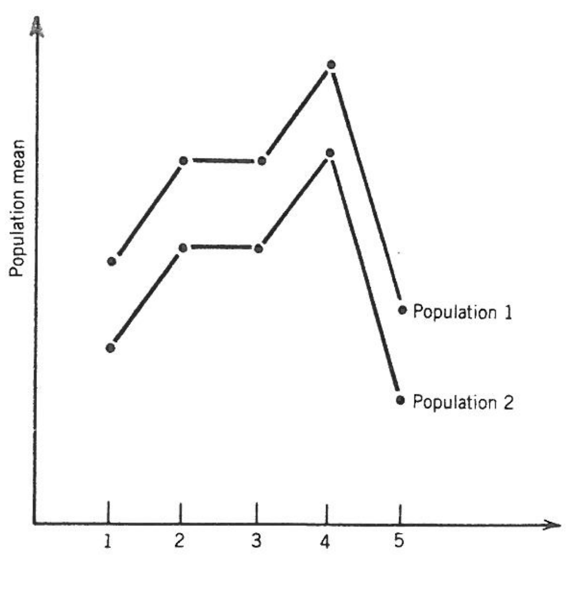

Fig. 3.2 Two parallel profiles

$$\left.\begin{array}{ll} C_1 \boldsymbol{v}_i \sim N_{d-1}(C_1 \boldsymbol{\eta}, C_1 \Sigma C_1^T), & i = 1, \dots, n_1 \\ C_1 \boldsymbol{w}_j \sim N_{d-1}(C_1 \boldsymbol{\nu}, C_1 \Sigma C_1^T), & j = 1, \dots, n_2 \end{array}\right\} \text{ all indep.}$$

H_{01} is equivalent to a test for equal means in two populations with equal variance matrices

Test statistic

$$T^2 = \frac{n_1 n_2}{n_1 + n_2} \left(C_1 (\bar{\boldsymbol{v}} - \bar{\boldsymbol{w}}) \right)^T \left(C_1 S_p C_1^T \right)^{-1} C_1 (\bar{\boldsymbol{v}} - \bar{\boldsymbol{w}})$$

$$\sim T^2(d-1, n_1+n_2-2) = \frac{(n_1+n_2-2)(d-1)}{n_1+n_2-d} F(d-1, n_1+n_2-d)$$

In practice we plot $(1, \bar{v}_1), \dots, (d, \bar{v}_d)$ and $(1, \bar{w}_1), \dots, (d, \bar{w}_d)$ in stead of the profiles

H_{02}: The profiles are at the same level

$$\Leftrightarrow \frac{1}{d}(\eta_1 + \eta_2 + \dots + \eta_d) = \frac{1}{d}(\nu_1 + \nu_2 + \dots + \nu_d)$$

$$\Leftrightarrow \boldsymbol{1}_d^T \boldsymbol{\eta} = \boldsymbol{1}_d^T \boldsymbol{\nu}$$

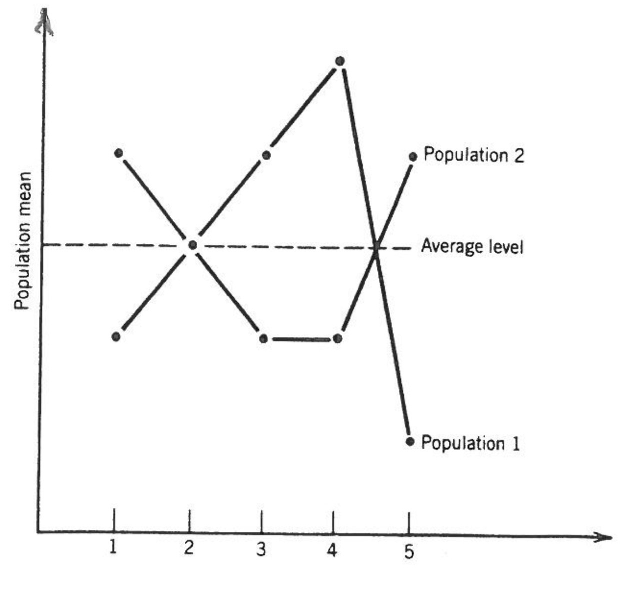

Fig. 3.3 Two profiles with the same average level.

$$\left.\begin{array}{ll} \underset{\sim}{1}_d^T \underset{\sim}{v}_i \sim N(\underset{\sim}{1}_d^T \underset{\sim}{\eta}, \underset{\sim}{1}_d^T \Sigma \underset{\sim}{1}_d), & i = 1, \dots, n_1 \\ \underset{\sim}{1}_d^T \underset{\sim}{w}_j \sim N(\underset{\sim}{1}_d^T \underset{\sim}{\nu}, \underset{\sim}{1}_d^T \Sigma \underset{\sim}{1}_d), & j = 1, \dots, n_2 \end{array}\right\} \text{ all indep.}$$

Test statistic for H_{02}:

$$t = \frac{\underset{\sim}{1}_d^T (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})}{\sqrt{\underset{\sim}{1}_d^T S_p \underset{\sim}{1}_d \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t(n_1 + n_2 - 2)$$

$H_{01} \cap H_{02}$: The two profiles coincide

H_{02} can be interpreted as no "main effect" from the population, but becomes difficult to interpret without preliminary acceptance of H_{01}.

H_{03}: No "main effect" from the d variables

$$\begin{aligned} &\Leftrightarrow \quad \tfrac{1}{2}(\eta_1 + \nu_1) = \tfrac{1}{2}(\eta_2 + \nu_2) = \dots = \tfrac{1}{2}(\eta_d + \nu_d) \\ &\Leftrightarrow \quad \eta_k + \nu_k - (\eta_{k-1} + \nu_{k-1}) = 0, \quad k = 2, \dots, d \\ &\Leftrightarrow \quad C_1(\underset{\sim}{\eta} + \underset{\sim}{\nu}) = \underset{\sim}{0} \end{aligned}$$

$H_{01} \cap H_{03}$: $\eta_1 = \eta_2 = \dots = \eta_d \wedge \nu_1 = \nu_2 = \dots = \nu_d$

$\Leftrightarrow$ The two profiles are parallel and horizontal lines

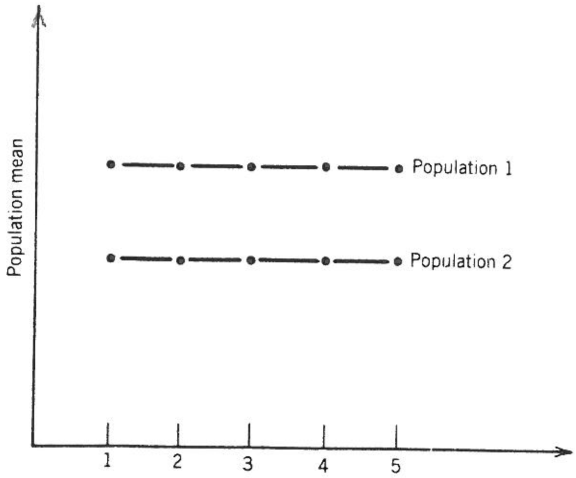

Fig. 3.4 Two parallel profiles with no effects due to the variables.

$H_{01} \cap H_{02} \cap H_{03}$: The two profiles are coinciding horizontal lines

Test of $H_{01} \cap H_{03}$

Consider $\underset{\sim}{\bar{x}} = \dfrac{n_1 \underset{\sim}{\bar{v}} + n_2 \underset{\sim}{\bar{w}}}{n_1 + n_2}$, $\underset{\sim}{\bar{x}} \sim N_d(\cdot, \cdot)$

$$E\underset{\sim}{\bar{x}} = \frac{n_1 E\underset{\sim}{\bar{v}} + n_2 E\underset{\sim}{\bar{w}}}{n_1 + n_2} = \frac{n_1 \underset{\sim}{\eta} + n_2 \underset{\sim}{\nu}}{n_1 + n_2}$$

$$\operatorname{Var} \underset{\sim}{\bar{x}} = \frac{n_1^2 \operatorname{Var} \underset{\sim}{\bar{v}} + n_2^2 \operatorname{Var} \underset{\sim}{\bar{w}}}{(n_1 + n_2)^2} = \frac{n_1^2 \frac{\Sigma}{n_1} + n_2^2 \frac{\Sigma}{n_2}}{(n_1 + n_2)^2} = \frac{\Sigma}{n_1 + n_2}$$

Now consider $C_1 \underset{\sim}{\bar{x}} \sim N_{d-1}(\cdot, \cdot)$

$$E[C_1 \underset{\sim}{\bar{x}}] = C_1 E \underset{\sim}{\bar{x}} = \frac{n_1 C_1 \underset{\sim}{\eta} + n_2 C_1 \underset{\sim}{\nu}}{n_1 + n_2}$$

When H_{01} : $C_1 \underset{\sim}{\eta} = C_1 \underset{\sim}{\nu}$ is true :

$$E[C_1 \underset{\sim}{\bar{x}}] = C_1 \underset{\sim}{\eta} = C_1 \underset{\sim}{\nu} = \tfrac{1}{2} C_1 (\underset{\sim}{\eta} + \underset{\sim}{\nu})$$

$$\operatorname{Var}(C_1 \underset{\sim}{\bar{x}}) = \frac{1}{n_1 + n_2} C_1 \Sigma C_1^T$$

When $H_{01} \cap H_{03}$ is true :

$$C_1 \underset{\sim}{\bar{x}} \sim N_{d-1}\left(\underset{\sim}{0}, \frac{1}{n_1 + n_2} C_1 \Sigma C_1^T\right)$$

Test statistic for $H_{01} \cap H_{03}$:

$$T^2 = (n_1 + n_2)(C_1 \underset{\sim}{\bar{x}})^T (C_1 S_p C_1^T)^{-1} C_1 \underset{\sim}{\bar{x}}$$

$$\sim T^2(d-1, n_1 + n_2 - 2) \quad \text{(as for test of } H_{01})$$

Ex.

EXAMPLE 3.8 A sample of 27 children aged about 8–9 years who had an inborn error of metabolism known as transient neonatal tyrosinemia (TNT) were compared with a closely matched sample of 27 normal children (called the control group) by their scores on the Illinois Test of Psycholingual Ability (ITPA). This test gives scores on 10 variables, namely;

$$x_1 = \text{auditory reception score},$$

$$x_2 = \text{visual reception score},$$

$$x_3 = \text{visual memory},$$

$$x_4 = \text{auditory association},$$

$$x_5 = \text{auditory memory},$$

$$x_6 = \text{visual association},$$

$$x_7 = \text{visual closure},$$

$$x_8 = \text{verbal expression},$$

$$x_9 = \text{grammatic closure},$$

$$x_{10} = \text{manual expression}.$$

The data are listed in Table 3.7 and the profiles for the sample means are given in Fig. 3.5. The standard deviation of each mean is $SD/\sqrt{27}$, or roughly 1.35 $(= 7/\sqrt{27})$, so that an inspection of Fig. 3.5 would suggest that the hypothesis of parallelism would not be rejected. The variable x_{10} is omitted from the following analysis.

cf. the book p. 122-123

The sample dispersion matrices ($\mathbf{S}_i = \mathbf{Q}_i/26$) for the control and TNT groups, respectively, are, to one decimal place,

$$\mathbf{S}_1 = \begin{pmatrix} 49.1 & 16.8 & 2.7 & 27.9 & 10.9 & 30.1 & 6.6 & 64.4 & 23.4 \\ - & 43.4 & 18.2 & 25.8 & 10.7 & 18.4 & 7.0 & 36.9 & 19.9 \\ - & - & 69.4 & 22.5 & 13.5 & 18.8 & 14.4 & 11.2 & 20.6 \\ - & - & - & 77.7 & 32.6 & 39.9 & 17.0 & 56.0 & 32.6 \\ - & - & - & - & 65.4 & 24.9 & 9.7 & 23.6 & 17.5 \\ - & - & - & - & - & 49.7 & 12.8 & 48.1 & 22.0 \\ - & - & - & - & - & - & 18.3 & 14.2 & 9.3 \\ - & - & - & - & - & - & - & 133.9 & 29.7 \\ - & - & - & - & - & - & - & - & 36.3 \end{pmatrix}$$

and

$$\mathbf{S}_2 = \begin{pmatrix} 56.6 & 21.4 & 8.6 & 32.8 & 9.6 & 15.3 & 18.9 & 26.1 & 10.3 \\ - & 65.4 & 8.1 & -1.7 & 17.1 & 16.1 & 14.9 & 11.3 & 16.4 \\ - & - & 30.3 & -0.2 & -0.5 & 3.0 & 5.6 & 10.2 & 10.8 \\ - & - & - & 75.5 & 14.1 & 7.0 & 18.7 & 35.7 & -2.7 \\ - & - & - & - & 55.1 & 7.3 & 14.4 & 27.4 & 6.2 \\ - & - & - & - & - & 42.3 & 5.5 & 4.1 & 10.4 \\ - & - & - & - & - & - & 22.7 & 24.3 & 13.3 \\ - & - & - & - & - & - & - & 82.2 & 19.9 \\ - & - & - & - & - & - & - & - & 44.1 \end{pmatrix}$$

As these two matrices are not too different and $n_1 = n_2 = 27$, we can assume that our test procedures are reasonably robust with regard to both nonnormality and moderate departures from $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ (see Section 3.6.2c). The pooled dispersion matrix $\mathbf{S}_p$ is the average of $\mathbf{S}_1$ and $\mathbf{S}_2$, and the T_0^2 statistic (3.100) for parallelism takes the value 2.2697. Hence, from (3.101),

$$F_0 = \frac{2.2697}{52}\frac{45}{8} = 0.2455 \sim F_{8,45},$$

and the test statistic is not significant, as $F_{8,45}^{0.05} = 2.2$. Forming the row sums of Table 3.7, we find from (3.102) that $t_0 = 1.88$. Since $t_{52}^{0.05} = 1.68$ and $t_{52}^{0.025} =$

2.01, we reject the hypothesis that the profiles are at the same level at the 10% level of significance, but not at the 5% level. Finally, the test statistic (3.103) takes the value $T_0^2 = 129.894$, so that

$$F_0 = \frac{129.894}{52}\frac{45}{8} = 14.05 \sim F_{8,45},$$

which is significant at the 1% level. We conclude that the scores on the d variables are different, which is to be expected from Fig. 3.5.

Summing up, we conclude that the two profiles are very similar. Although there is a positive difference for all the variables (Fig. 3.5), this difference is not great when compared to a standard deviation of about 1.4 for each mean. It should be noted that these differences have high positive correlations, which could be a contributing factor to the systematic difference.

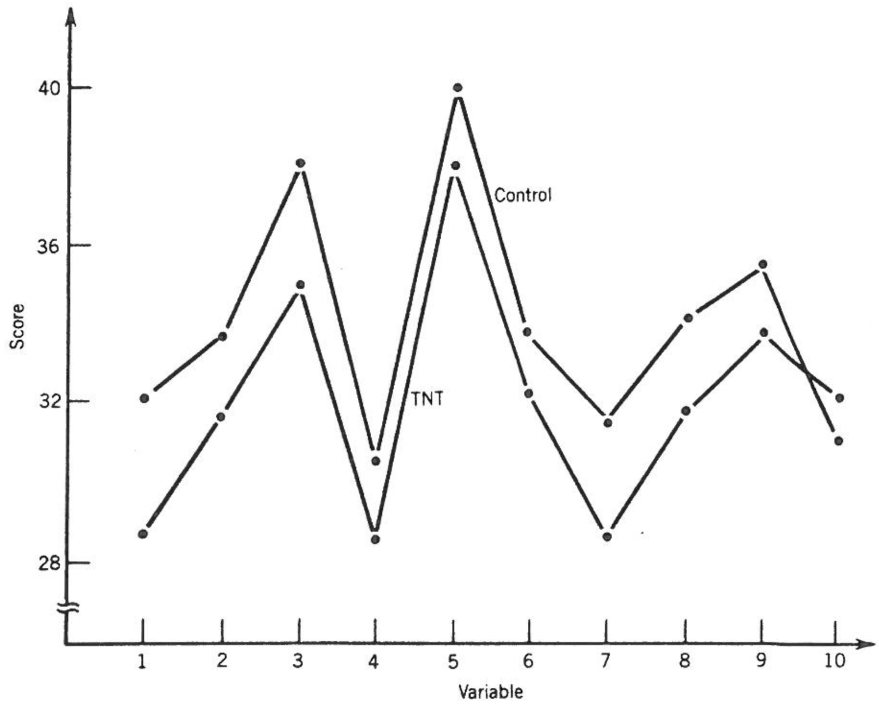

Fig. 3.5 Profiles of average scores in 10 categories for normal children (control) and children with transient neonatal tyrosinemia (TNT).