Hypothesis testing – preliminaries

Model: $Y = XB + U$, X $n \times p$, $\operatorname{rank} X = r \le p$, $n - r \ge d$

$\underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \ldots, n$, independent

H_0: $AB = C$, $\operatorname{rank} A = q \le r$, $\operatorname{tr}(AB)$ estimable i.e. $A = MX$

H_0 is equivalent to H_{0j}: $A\underset{\sim}{\beta}^{(j)} = \underset{\sim}{c}^{(j)}$, $j = 1, \ldots, d$

$E = (Y - X\hat{B})^T(Y - X\hat{B}) = Y^T(I_n - P_\Omega)Y = U^T(I_n - P_\Omega)U$

$= (\tilde{Y} + XB_0)^T(I_n - P_\Omega)(\tilde{Y} + XB_0) = \tilde{Y}^T(I_n - P_\Omega)\tilde{Y}$

$E_H = (Y - X\hat{B}_H)^T(Y - X\hat{B}_H) = \tilde{Y}^T(I_n - P_\omega)\tilde{Y}$

$E_H = U^T(I_n - P_\omega)U$ when H_0 is true cf. lecture note MA 13 p. 3

$H = E_H - E = \tilde{Y}^T(P_\Omega - P_\omega)\tilde{Y} = \tilde{Y}^T P_{\omega^\perp \cap \Omega}\tilde{Y}$

$H = U^T(P_\Omega - P_\omega)U = U^T P_{\omega^\perp \cap \Omega} U$ when H_0 is true

Theorem 8.5

$$\left.\begin{array}{l} E \sim W_d(n-r, \Sigma) \\ H \sim W_d(q, \Sigma) \quad \text{when } H_0 \text{ is true} \end{array}\right\} \text{independent}$$

Proof:

$E = U^T(I_n - P_\Omega)U \sim W_d(n-r, \Sigma)$ cf. corollary 1 to theorem 2.4

$H = U^T(P_\Omega - P_\omega)U \sim W_d(q, \Sigma)$ – – – – – – ,

as $\operatorname{rank}(P_\Omega - P_\omega) = \operatorname{rank} P_{\omega^\perp \cap \Omega} = \operatorname{rank} A = q$

$(P_{\omega^\perp \cap \Omega} = X(X^TX)^- A^T(A(X^TX)^-A^T)^{-1}A(X^TX)^-X^T)$

$(I_n - P_\Omega)(P_\Omega - P_\omega) = P_\Omega - P_\omega - P_\Omega + P_\omega = 0$

$\Rightarrow$ E and H independent, cf. corollary 2 to theorem 2.4

Note that E and H are also independent when H_1 is true as $H \sim W_d(q, \Sigma; \Delta)$ and the corollaries to theorem 2.4 still are valid.

"Corollary" 1 : $H = (A\hat{B} - C)^T (A(X^TX)^- A^T)^{-1} (A\hat{B} - C)$

Proof : $H = \tilde{Y}^T P_{\omega^\perp \cap \Omega} \tilde{Y} = (Y - XB_0)^T P_{\omega^\perp \cap \Omega} (Y - XB_0)$

$= (Y - XB_0)^T X (X^TX)^- A^T (A(X^TX)^- A^T)^{-1} A (X^TX)^- X^T (Y - XB_0)$

$= (A(X^TX)^- X^T Y - MX(X^TX)^- X^T X B_0)^T (A(X^TX)^- A^T)^{-1}$
$(A(X^TX)^- X^T Y - MX(X^TX)^- X^T X B_0)$

$= (A\hat{B} - MP_\Omega X B_0)^T (A(X^TX)^- A^T)^{-1} (A\hat{B} - MP_\Omega X B_0)$

$= (A\hat{B} - MXB_0)^T (A(X^TX)^- A^T)^{-1} (A\hat{B} - MXB_0)$

$= (A\hat{B} - AB_0)^T (A(X^TX)^- A^T)^{-1} (A\hat{B} - AB_0)$

$= (A\hat{B} - C)^T (A(X^TX)^- A^T)^{-1} (A\hat{B} - C)$

"Corollary" 2 : $E[H] = q\Sigma + D$ when H_1 is true

$H \sim W_d(q, \Sigma; \Sigma^{-\frac{1}{2}} D \Sigma^{-\frac{1}{2}})$ when H_1 is true

$D = (AB - C)^T (A(X^TX)^- A^T)^{-1} (AB - C)$

Proof : $E[H] = E[\tilde{Y}^T P_{\omega^\perp \cap \Omega} \tilde{Y}] = E[(Y - XB_0)^T P_{\omega^\perp \cap \Omega} (Y - XB_0)]$

$= (\operatorname{tr} P_{\omega^\perp \cap \Omega}) \operatorname{Var}(\underset{\sim}{y}_i - B_0^T \underset{\sim}{x}_i)$

$+ (EY - XB_0)^T X (X^TX)^- A^T (A(X^TX)^- A^T)^{-1} A (X^TX)^- X^T (EY - XB_0)$

cf. corollary to lemma 1.1

$= q \operatorname{Var} \underset{\sim}{y}_i + (AB - C)^T (A(X^TX)^- A^T)^{-1} (AB - C)$ *

$= q\Sigma + D$

$H \sim W_d(q, \Sigma; \Sigma^{-\frac{1}{2}} D \Sigma^{-\frac{1}{2}})$ cf. definition of the non central Wishart distribution

Note that $D \geq 0$ with equality only for $AB = C$

thus $D > 0$ when H_1 is true

* as $A(X^TX)^- X^T EY = MX(X^TX)^- X^T EY = MP_\Omega \Theta = M\Theta = MXB = AB$

and $A(X^TX)^- X^T X B_0 = MX(X^TX)^- X^T X B_0 = MP_\Omega X B_0 = MXB_0 = AB_0 = C$

Test theory

Model: $Y = XB + U$, $\operatorname{rank} X = r \le p$, $\underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, n$ independent

Test $H_0: AB = C$, $H_1: AB \neq C$, $\operatorname{rank} A = q \le p$

From theorem 8.5 and "corollaries"

$$\left.\begin{aligned} E[E] &= (n-r)\Sigma \\ E[H] &= q\Sigma + D \end{aligned}\right\} \quad E \text{ and } H \text{ independent}$$

$D = 0$ when H_0 is true, $D > 0$ when H_1 is true

U-∩-test

$\forall \underset{\sim}{\ell} \neq \underset{\sim}{0}$: $Y\underset{\sim}{\ell} = XB\underset{\sim}{\ell} + U\underset{\sim}{\ell}$, $U\underset{\sim}{\ell} \sim N_n(\underset{\sim}{0}, \sigma_\ell^2 I_n)$

cf. lemma 2.3 (iv)

Let $Y\underset{\sim}{\ell} = \underset{\sim}{y}$, $B\underset{\sim}{\ell} = \underset{\sim}{\beta}$, $U\underset{\sim}{\ell} = \underset{\sim}{u}$, also $C\underset{\sim}{\ell} = \underset{\sim}{c}$

hence $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{u}$, $u_i \sim N(0, \sigma_\ell^2)$, $i = 1, \dots, n$, indep.

$H_{0\ell}: A\underset{\sim}{\beta} = \underset{\sim}{c}$, $H_{1\ell}: A\underset{\sim}{\beta} \neq \underset{\sim}{c}$

H_0 is equivalent to $\bigcap_\ell H_{0\ell}$

Test statistic for $H_{0\ell}$:

$$F_\ell = \frac{n-r}{q} \frac{(A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})^T (A(X^TX)^-A^T)^{-1} (A\hat{\underset{\sim}{\beta}} - \underset{\sim}{c})}{\| \underset{\sim}{y} - X\hat{\underset{\sim}{\beta}} \|^2} \sim F(q, n-r) \text{ when } H_{0\ell} \text{ is true}$$

($A\underset{\sim}{\beta}$ assumed estimabel)

$$F_\ell = \frac{n-r}{q} \frac{\underset{\sim}{\ell}^T (A\hat{B} - C)^T (A(X^TX)^-A^T)^{-1} (A\hat{B} - C)\underset{\sim}{\ell}}{\underset{\sim}{\ell}^T (Y - X\hat{B})^T (Y - X\hat{B})\underset{\sim}{\ell}}$$

$$= \frac{n-r}{q} \frac{\underset{\sim}{\ell}^T H \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T E \underset{\sim}{\ell}}, \quad \text{cf. "corollary" 1 to theorem 8.5}$$

Hence

$$F_\ell = \frac{n-r}{q} \frac{\underset{\sim}{\ell}^T H \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T E \underset{\sim}{\ell}} \sim F(q, n-r) \quad \text{when } H_{0\ell} \text{ is true}$$

Acceptance area for H_0:

$$\bigcap_\ell \{ Y \mid F_\ell \le F_{1-\alpha}(q,n-r) \} = \{ Y \mid \sup_{\underset{\sim}{\ell}} \frac{\underset{\sim}{\ell}^T H \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T E \underset{\sim}{\ell}} \le \frac{q}{n-r} F_{1-\alpha}(q,n-r) \}$$

$$= \{ Y \mid \varphi_{max} \le \frac{q}{n-r} F_{1-\alpha}(q,n-r) \}$$

cf. A7.5

φ_{max} is the greatest eigenvalue corresponding to HE^{-1}

$\theta_{max} = \dfrac{\varphi_{max}}{1+\varphi_{max}}$ is the greatest eigenvalue corresponding to $H(E+H)^{-1}$, cf. formula (2.27) and (2.28)

Table D14 is based on θ_{max}

Likelihood ratio test

$$\ell = \frac{L(\hat{\Theta}_H, \hat{\Sigma}_H)}{L(\hat{\Theta}, \hat{\Sigma})} = \frac{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma}_H)^{-\frac{n}{2}}}{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma})^{-\frac{n}{2}}} = \left(\frac{\det \hat{\Sigma}}{\det \hat{\Sigma}_H} \right)^{\frac{n}{2}}$$

$$\Lambda = \ell^{\frac{2}{n}} = \frac{n^{-d} \det E}{n^{-d} \det E_H} = \frac{\det E}{\det(E+H)} \sim U(d,q,n-r)$$

when H_0 is true

Small values of Λ are critical for the hypothesis

cf formula (2.40) etc.

$$P\left(-f \ln \Lambda \le C_\alpha \chi^2_{1-\alpha}(dq)\right) = 1-\alpha, \quad f = n-r-\tfrac{1}{2}(d-q+1)$$

cf. formula (2.46), C_α is tabled (table D13)

Simultaneous confidence intervals

Assume rank $X = p$

First consider the hypothesis $H_0 : AB = 0$ again

With $\underset{\sim}{b} := \underset{\sim}{e}$ we have $\underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{u}$, $u_i \sim N(0, \sigma_b^2)$

$H_{0b} : A\underset{\sim}{\beta} = \underset{\sim}{0}$, $H_{1b} : A\underset{\sim}{\beta} \neq \underset{\sim}{0}$

Let $\underset{\sim}{a}^T A = \underset{\sim}{a}_0^T$ for all $\underset{\sim}{a}$

Consider H_{0ab} : $\underset{\sim}{a}^T AB\underset{\sim}{b} = 0 \iff \underset{\sim}{a}_0^T \underset{\sim}{\beta} = 0$

$\bigcap\limits_a \bigcap\limits_b H_{0ab}$ is equivalent to H_0

Test statistic for test of H_{0ab} cf. lecture note MA 12 p. 5 :

$$F_{ab} = \frac{n-p}{1} \frac{(\underset{\sim}{a}_0^T\hat{\underset{\sim}{\beta}})^T(\underset{\sim}{a}_0^T(X^TX)^{-1}\underset{\sim}{a}_0)^{-1}\underset{\sim}{a}_0^T\hat{\underset{\sim}{\beta}}}{\|\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}\|^2} \sim F(1, n-p)$$

when H_{0ab} is true

$$\hat{\underset{\sim}{\beta}} = (X^TX)^{-1}X^T\underset{\sim}{y}$$

$$\|\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}\|^2 = \|\underset{\sim}{y} - P_\Omega \underset{\sim}{y}\|^2 = \underset{\sim}{y}^T(I_n - P_\Omega)\underset{\sim}{y} = \underset{\sim}{b}^T Y^T(I_n - P_\Omega)Y\underset{\sim}{b}$$
$$= \underset{\sim}{b}^T E \underset{\sim}{b}$$

$$F_{ab} = \frac{(n-p)(\underset{\sim}{a}_0^T\hat{\underset{\sim}{\beta}})^2}{\underset{\sim}{a}_0^T(X^TX)^{-1}\underset{\sim}{a}_0 \; \underset{\sim}{b}^T E \underset{\sim}{b}} = \frac{(n-p)(\underset{\sim}{a}^T A\hat{B}\underset{\sim}{b})^2}{\underset{\sim}{a}^T A(X^TX)^{-1}A^T\underset{\sim}{a} \; \underset{\sim}{b}^T E \underset{\sim}{b}}$$

Alternative calculation of F_{ab} :

$$\underset{\sim}{a}_0^T\hat{\underset{\sim}{\beta}} = \underset{\sim}{a}_0^T(X^TX)^{-1}X^T\underset{\sim}{y}$$

$$\mathrm{Var}(\underset{\sim}{a}_0^T\hat{\underset{\sim}{\beta}}) = \underset{\sim}{a}_0^T(X^TX)^{-1}X^T \sigma_b^2 I_n X(X^TX)^{-1}\underset{\sim}{a}_0$$
$$= \underset{\sim}{a}_0^T(X^TX)^{-1}\underset{\sim}{a}_0 \, \sigma_b^2$$

$$\underset{\sim}{a}_o^T \hat{\underset{\sim}{\beta}} \sim N\left(\underset{\sim}{a}_o^T \underset{\sim}{\beta},\ \underset{\sim}{a}_o^T (X^T X)^{-1} \underset{\sim}{a}_o\, \sigma_b^2\right)$$

σ_b^2 is estimated by $\underset{\sim}{b}^T S \underset{\sim}{b}$, $S = \dfrac{E}{n-r}$

$$\frac{\underset{\sim}{a}_o^T \hat{\underset{\sim}{\beta}} - 0}{\sqrt{\underset{\sim}{a}_o^T (X^T X)^{-1} \underset{\sim}{a}_o\, \underset{\sim}{b}^T \frac{E}{n-r} \underset{\sim}{b}}} \sim t(n-r) \text{ when } H_{0ab} \text{ is true}$$

$$\frac{(n-r)(\underset{\sim}{a}_o^T \hat{\underset{\sim}{\beta}})^2}{\underset{\sim}{a}_o^T (X^T X)^{-1} \underset{\sim}{a}_o\, \underset{\sim}{b}^T E \underset{\sim}{b}} \sim F(1, n-r) \text{ when } H_{0ab} \text{ is true}$$

hence the same F_{ab} as before

Acceptance area for H_0 :

$$\bigcap_a \bigcap_b \left\{ Y \,\middle|\, \frac{(n-r)(\underset{\sim}{a}_o^T \hat{\underset{\sim}{\beta}})^2}{\underset{\sim}{a}_o^T (X^T X)^{-1} \underset{\sim}{a}_o\, \underset{\sim}{b}^T E \underset{\sim}{b}} \le F_{1-\alpha}(1, n-r) \right\}$$

$$= \left\{ Y \,\middle|\, \sup_{\underset{\sim}{a}, \underset{\sim}{b}} \frac{(\underset{\sim}{a}^T A \hat{B} \underset{\sim}{b})^2}{\underset{\sim}{a}^T A (X^T X)^{-1} A^T \underset{\sim}{a}\, \underset{\sim}{b}^T E \underset{\sim}{b}} \le \frac{1}{n-r} F_{1-\alpha}(1, n-r) \right\}$$

$$= \left\{ Y \,\middle|\, \varphi_{max} \le \frac{1}{n-r} F_{1-\alpha}(1, n-r) \right\},$$

where φ_{max} is the greatest eigenvalue corresponding to $(A\hat{B})^T (A(X^T X)^{-1} A^T)^{-1} A\hat{B} E^{-1} = HE^{-1}$, cf. A 7.7 and A 1.4 , hence the same test statistic here as we found in section 8.2.6 a

Whether H_{0ab} is true or not we have

$$\frac{(n-r)(\underset{\sim}{a}_o^T \hat{\underset{\sim}{\beta}} - \underset{\sim}{a}_o^T \underset{\sim}{\beta})^2}{\underset{\sim}{a}_o^T (X^T X)^{-1} \underset{\sim}{a}_o\, \underset{\sim}{b}^T E \underset{\sim}{b}} = \frac{(n-r)(\underset{\sim}{a}^T A \hat{B} \underset{\sim}{b} - \underset{\sim}{a}^T A B \underset{\sim}{b})^2}{\underset{\sim}{a}^T A (X^T X)^{-1} A^T \underset{\sim}{a}\, \underset{\sim}{b}^T E \underset{\sim}{b}}$$

$$= \frac{(n-r)(\underset{\sim}{a}^T (A\hat{B} - AB) \underset{\sim}{b})^2}{\underset{\sim}{a}^T A (X^T X)^{-1} A^T \underset{\sim}{a}\, \underset{\sim}{b}^T E \underset{\sim}{b}} \sim F(1, n-r) \quad , \text{ as } E\hat{\underset{\sim}{\beta}} = \underset{\sim}{\beta}$$

$$\Rightarrow \sup_{\underset{\sim}{a},\underset{\sim}{b}} \frac{(\underset{\sim}{a}^T(A\hat{B}-AB)\underset{\sim}{b})^2}{\underset{\sim}{a}^T A(X^TX)^{-1}A^T\underset{\sim}{a}\ \underset{\sim}{b}^T E \underset{\sim}{b}} = \varphi_{max}$$

where φ_{max} is the greatest eigenvalue corresponding to $(A\hat{B}-AB)^T(A(X^TX)^{-1}A^T)^{-1}(A\hat{B}-AB)E^{-1}$ cf. A 7.7 and A1.4

Choose $\varphi_{1-\alpha}$ so $P(\varphi_{max} \le \varphi_{1-\alpha}) = 1-\alpha$

Hence confidence intervals for $\underset{\sim}{a}^T AB\underset{\sim}{b}$ with simultaneous confidence level $1-\alpha$:

$$\underset{\sim}{a}^T AB\underset{\sim}{b} = \underset{\sim}{a}^T A\hat{B}\underset{\sim}{b} \pm \sqrt{\varphi_{1-\alpha}\underset{\sim}{a}^T A(X^TX)^{-1}A^T\underset{\sim}{a}\ \underset{\sim}{b}^T E\underset{\sim}{b}}$$

in particular when $A = I_p$ (now $E \sim W_d(p, \Sigma)$):

$$\underset{\sim}{a}^T B\underset{\sim}{b} = \underset{\sim}{a}^T \hat{B}\underset{\sim}{b} \pm \sqrt{\varphi_{1-\alpha}\underset{\sim}{a}^T (X^TX)^{-1}\underset{\sim}{a}\ \underset{\sim}{b}^T E\underset{\sim}{b}}$$

These intervals include all β_{ij} (choose suitable $\underset{\sim}{a}$ and $\underset{\sim}{b}$)

Bonferroni intervals (m intervals):

$$\underset{\sim}{a}^T B\underset{\sim}{b} = \underset{\sim}{a}^T \hat{B}\underset{\sim}{b} \pm t_{1-\frac{\alpha}{2m}}(n-p)\sqrt{\underset{\sim}{a}^T (X^TX)^{-1}\underset{\sim}{a}\ \underset{\sim}{b}^T E\underset{\sim}{b}}$$

cf. ex. 8.9

Test of equal means assuming equal variance matrices

(earlier dealt with, see lecture note MA 11)

$$\left.\begin{array}{ll} \underset{\sim}{v}_i \sim N_d(\underset{\sim}{\mu}_1, \Sigma), & i = 1, \dots, n_1 \\ \underset{\sim}{w}_j \sim N_d(\underset{\sim}{\mu}_2, \Sigma), & j = 1, \dots, n_2 \end{array}\right\} \text{ all independent}$$

Let $\underset{\sim}{y}_i = \underset{\sim}{v}_i \;, \quad i = 1, \dots, n_1$

$\underset{\sim}{y}_i = \underset{\sim}{w}_{i-n_1} \;, \quad i = n_1+1, \dots, n_1+n_2$

hence $Y = \begin{bmatrix} V \\ W \end{bmatrix}$, $\quad EY = \begin{bmatrix} \underset{\sim}{1}_{n_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n_2} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\mu}_1^T \\ \underset{\sim}{\mu}_2^T \end{bmatrix} = XB$

$$Y = XB + U \;, \quad \underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma) \;, \quad i = 1, \dots, n_1+n_2$$

independent

$$H_0: \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 \iff \underset{\sim}{\mu}_1^T - \underset{\sim}{\mu}_2^T = \underset{\sim}{0}^T \iff [1 \;\; -1]\, B = \underset{\sim}{0}^T \iff AB = C$$

$$\hat{B} = (X^T X)^{-1} X^T Y = \left(\begin{bmatrix} \underset{\sim}{1}_{n_1}^T & \underset{\sim}{0}^T \\ \underset{\sim}{0}^T & \underset{\sim}{1}_{n_2}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{1}_{n_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n_2} \end{bmatrix} \right)^{-1} \begin{bmatrix} \underset{\sim}{1}_{n_1}^T & \underset{\sim}{0}^T \\ \underset{\sim}{0}^T & \underset{\sim}{1}_{n_2}^T \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_i \underset{\sim}{v}_i^T \\ \sum_j \underset{\sim}{w}_j^T \end{bmatrix} = \begin{bmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} n_1 \bar{\underset{\sim}{v}}^T \\ n_2 \bar{\underset{\sim}{w}}^T \end{bmatrix} = \begin{bmatrix} \bar{\underset{\sim}{v}}^T \\ \bar{\underset{\sim}{w}}^T \end{bmatrix}$$

$$Y - X\hat{B} = \begin{bmatrix} V \\ W \end{bmatrix} - \begin{bmatrix} \underset{\sim}{1}_{n_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{1}_{n_2} \end{bmatrix} \begin{bmatrix} \bar{\underset{\sim}{v}}^T \\ \bar{\underset{\sim}{w}}^T \end{bmatrix} = \begin{bmatrix} V - \underset{\sim}{1}_{n_1} \bar{\underset{\sim}{v}}^T \\ W - \underset{\sim}{1}_{n_2} \bar{\underset{\sim}{w}}^T \end{bmatrix}$$

$$E = (Y - X\hat{B})^T (Y - X\hat{B}) = \begin{bmatrix} V - \underset{\sim}{1}_{n_1} \bar{\underset{\sim}{v}}^T \\ W - \underset{\sim}{1}_{n_2} \bar{\underset{\sim}{w}}^T \end{bmatrix}^T \begin{bmatrix} V - \underset{\sim}{1}_{n_1} \bar{\underset{\sim}{v}}^T \\ W - \underset{\sim}{1}_{n_2} \bar{\underset{\sim}{w}}^T \end{bmatrix}$$

$$= \left[\underset{\sim}{v}_1 - \bar{\underset{\sim}{v}} \; \cdots \; \underset{\sim}{v}_{n_1} - \bar{\underset{\sim}{v}} \quad \underset{\sim}{w}_1 - \bar{\underset{\sim}{w}} \; \cdots \; \underset{\sim}{w}_{n_2} - \bar{\underset{\sim}{w}} \right] \begin{bmatrix} \underset{\sim}{v}_1^T - \bar{\underset{\sim}{v}}^T \\ \vdots \\ \underset{\sim}{v}_{n_1}^T - \bar{\underset{\sim}{v}}^T \\ \underset{\sim}{w}_1^T - \bar{\underset{\sim}{w}}^T \\ \vdots \\ \underset{\sim}{w}_{n_2}^T - \bar{\underset{\sim}{w}}^T \end{bmatrix}$$

$$= \sum_{i=1}^{n_1} (\underset{\sim}{v}_i - \bar{\underset{\sim}{v}})(\underset{\sim}{v}_i - \bar{\underset{\sim}{v}})^T + \sum_{j=1}^{n_2} (\underset{\sim}{w}_j - \bar{\underset{\sim}{w}})(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}})^T$$

$$= Q_1 + Q_2 = (n_1 - 1) S_1 + (n_2 - 1) S_2 = (n_1 + n_2 - 2) S_p$$

$$A\hat{B} = [1 \;\; -1] \begin{bmatrix} \bar{\underset{\sim}{v}}^T \\ \bar{\underset{\sim}{w}}^T \end{bmatrix} = \bar{\underset{\sim}{v}}^T - \bar{\underset{\sim}{w}}^T = (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T$$

$$H = (A\hat{B})^T (A (X^T X)^{-1} A^T)^{-1} A\hat{B}$$ cf. "corollary" 1 to theorem 8.5

$$= (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}}) \left([1 \;\; -1] \begin{bmatrix} \frac{1}{n_1} & 0 \\ 0 & \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T$$

$$= (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}}) \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T$$

$$= \frac{n_1 n_2}{n_1 + n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T, \quad \text{rank } H = 1$$

$$T_g^2 = m_E \operatorname{tr}(H E^{-1}) \sim T^2(d, m_E)$$ cf. formula (2.56)

$$T_g^2 = (n_1 + n_2 - 2) \operatorname{Tr} \left(\frac{n_1 n_2}{n_1 + n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T \frac{1}{n_1 + n_2 - 2} S_p^{-1} \right)$$

$$= \frac{n_1 n_2}{n_1 + n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T S_p^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}}) \sim T^2(d, n_1 + n_2 - 2)$$

hence the same test statistic as found earlier in section 3.6.2