Wishart distribution

W $d \times d$ stoch. matrix, $W > 0$ a.s.

m is an integer, $m \geq d$

Σ $d \times d$ matrix, $\Sigma > 0$

$\frac{d(d+1)}{2}$ variables

$$\begin{bmatrix} w_{11} & \cdots & w_{1d} \\ \vdots & & \\ w_{d1} & \cdots & w_{dd} \end{bmatrix}$$

Def. a $\quad W \sim W_d(m, \Sigma)$

$\Leftrightarrow \quad f(w_{11}, \ldots, w_{dd}) = \frac{1}{c} (\det W)^{\frac{m-d-1}{2}} \operatorname{etr}\left(-\frac{1}{2} \Sigma^{-1} W\right)$, where

$$c = 2^{\frac{md}{2}} (\det \Sigma)^{\frac{m}{2}} \Gamma_d\left(\frac{m}{2}\right) \quad \text{and}$$

$$\Gamma_d\left(\frac{m}{2}\right) = \pi^{\frac{d(d-1)}{4}} \prod_{j=1}^{d} \Gamma\left(\frac{m+1-j}{2}\right)$$

$\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_m$ independent, $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$

Def. b $\quad W \sim W_d(m, \Sigma) \Leftrightarrow W = \sum_{i=1}^{m} \underset{\sim}{x}_i \underset{\sim}{x}_i^T$

Note: $\quad W = [\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_m] \begin{bmatrix} \underset{\sim}{x}_1^T \\ \vdots \\ \underset{\sim}{x}_m^T \end{bmatrix} = X^T X$

Def. b can be enhanced to include singular Wishart distributions, i.e. $m < d$ and/or $\Sigma \geq 0$.

Theorem 2.2: $W \sim W_d(m, \Sigma)$, C $q \times d$, rank $C = q$

$$\Rightarrow \quad C W C^T \sim W_q(m, C \Sigma C^T)$$

Proof : Assume $W_0 = \sum_{i=1}^{m} \underset{\sim}{x}_i \underset{\sim}{x}_i^T$ where $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$ and $\underset{\sim}{x}_i$'s independent

$\Rightarrow W_0 \sim W$ ($\Rightarrow$: W_0 and W have the same distribution)

$$C W_0 C^T = \sum_i C \underset{\sim}{x}_i \underset{\sim}{x}_i^T C^T = \sum_i (C\underset{\sim}{x}_i)(C\underset{\sim}{x}_i)^T \sim W_q(m, C\Sigma C^T)$$

as $C\underset{\sim}{x}_i \sim N_q(\underset{\sim}{0}, C\Sigma C^T)$ cf. th. 2.1 (i)

$$C W_0 C^T \sim W_q(m, C\Sigma C^T) \Rightarrow C W C^T \sim W_q(m, C\Sigma C^T)$$

Corollary 1 :

$C = \underset{\sim}{l}^T \Rightarrow \underset{\sim}{l}^T W_0 \underset{\sim}{l} = \sum_i (\underset{\sim}{l}^T \underset{\sim}{x}_i)^2 \sim \sigma_l^2 \chi^2(m)$ as

$\underset{\sim}{l}^T \underset{\sim}{x}_i \sim N(0, \sigma_l^2)$, $\sigma_l^2 = \underset{\sim}{l}^T \Sigma \underset{\sim}{l}$

$$\underset{\sim}{l}^T W_0 \underset{\sim}{l} \sim \sigma_l^2 \chi^2(m) \Rightarrow \underset{\sim}{l}^T W \underset{\sim}{l} \sim \sigma_l^2 \chi^2(m)$$

Corollary 2 :

$$\underset{\sim}{l}^T = [0 \dots 0 1 0 \dots 0] \Rightarrow \underset{\sim}{l}^T W \underset{\sim}{l} = w_{jj}$$

$$\Rightarrow w_{jj} \sim \sigma_j^2 \chi^2(m)$$

Note : w_{jk}, $k \neq j$, is not χ^2 distributed

Lemma 2.3 $\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_m$ indep., $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$

(i) $\underset{\sim}{x}^{(j)} \sim N_m(\underset{\sim}{0}, \sigma_j^2 I_m)$

(ii) $X^T \underset{\sim}{a} \sim N_d(\underset{\sim}{0}, \|\underset{\sim}{a}\|^2 \Sigma)$

(iii) $\underset{\sim}{a}_1, \dots, \underset{\sim}{a}_r$ mutual orthogonal, $r \leq m$
$\Rightarrow X^T \underset{\sim}{a}_1, \dots, X^T \underset{\sim}{a}_r$ independent

(iv) $X \underset{\sim}{b} \sim N_m(\underset{\sim}{0}, \sigma_b^2 I_m)$, $\sigma_b^2 = \underset{\sim}{b}^T \Sigma \underset{\sim}{b}$

Proof :

ad (i) $x_{ij} \sim N(0, \sigma_j^2) \Rightarrow \underset{\sim}{x}^{(j)} \sim N_m(\underset{\sim}{0}, \sigma_j^2 I_m)$

ad (ii) $X^T \underset{\sim}{a} = \sum_k a_k \underset{\sim}{x_k} \sim N_d(\underset{\sim}{0}, \sum_k a_k^2 \Sigma)$ cf. ex. 2.6

$$= N_d(\underset{\sim}{0}, \|\underset{\sim}{a}\|^2 \Sigma)$$

ad (iii) Let $\underset{\sim}{u_i} = X^T \underset{\sim}{a_i}$, $i = 1, \dots, r$

$$Cov(\underset{\sim}{u_i}, \underset{\sim}{u_j}) = Cov\left(\sum_k a_{ik} \underset{\sim}{x_k}, \sum_l a_{jl} \underset{\sim}{x_l}\right)$$

$$= \sum_k \sum_l a_{ik} a_{jl} \, Cov(\underset{\sim}{x_k}, \underset{\sim}{x_l})$$

$$= \sum_k a_{ik} a_{jk} \, Var \, \underset{\sim}{x_k} + n(n-1) \, 0$$

$$= \underset{\sim}{a_i}^T \underset{\sim}{a_j} \, \Sigma = 0 \text{ when } i \neq j$$

$$\underset{\sim}{u_i} = \sum_k a_{ik} \underset{\sim}{x_k} = \sum_k a_{ik} I_d \underset{\sim}{x_k}$$

$$= \left[a_{i1} I_d \; \dots \; a_{im} I_d \right] \begin{bmatrix} \underset{\sim}{x_1} \\ \vdots \\ \underset{\sim}{x_m} \end{bmatrix}$$

$$= A_i \underset{\sim}{x} \, , \text{ where}$$

$$\underset{\sim}{x} \sim N_{md}(\underset{\sim}{0}, I_m \otimes \Sigma)$$

$\Rightarrow \underset{\sim}{u_i}$'s independent, cf. th. 2.1 (v)

ad (iv) $X \underset{\sim}{b} = \begin{bmatrix} \underset{\sim}{x_1}^T \underset{\sim}{b} \\ \vdots \\ \underset{\sim}{x_m}^T \underset{\sim}{b} \end{bmatrix}$, $\underset{\sim}{x_i}^T \underset{\sim}{b} = \underset{\sim}{b}^T \underset{\sim}{x_i} \sim N(0, \sigma_b^2)$,

$$\sigma_b^2 = \underset{\sim}{b}^T \Sigma \underset{\sim}{b}$$

$$\Rightarrow X \underset{\sim}{b} \sim N_m(\underset{\sim}{0}, \sigma_b^2 I_m)$$

Corollary to (ii) :

$$\underset{\sim}{a} = \frac{1}{m} \underset{\sim}{1_m} \Rightarrow X^T \underset{\sim}{a} = \bar{\underset{\sim}{x}} \sim N_d\left(\underset{\sim}{0}, \frac{\Sigma}{m}\right)$$

Theorem 2.4 : $X = [\underset{\sim}{x}_1 \dots \underset{\sim}{x}_m]^T$, $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, n$, indep.

$\forall \underset{\sim}{\ell} \neq \underset{\sim}{0}$: $\underset{\sim}{y} = X\underset{\sim}{\ell} \sim N_m(\underset{\sim}{0}, \sigma_\ell^2 I_m)$, $\sigma_\ell^2 = \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}$, cf. lemma 2.3 (iv)

A , B <u>sym.</u> $m \times m$ matrices , rank A = r , rank B = s

$\underset{\sim}{b}$ m-dim. vector

(i) $X^T A X \sim W_d(r, \Sigma) \iff \forall \underset{\sim}{\ell} : \underset{\sim}{y}^T A \underset{\sim}{y} \sim \sigma_\ell^2 \chi^2(r)$, $\sigma_\ell^2 = \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}$

(ii) $\left.\begin{array}{l} X^T A X \sim W_d(r, \Sigma) \\ X^T B X \sim W_d(s, \Sigma) \end{array}\right\}$ independent

$\iff \forall \underset{\sim}{\ell} : \left\{\begin{array}{l} \underset{\sim}{y}^T A \underset{\sim}{y} \sim \sigma_\ell^2 \chi^2(r) \\ \underset{\sim}{y}^T B \underset{\sim}{y} \sim \sigma_\ell^2 \chi^2(s) \end{array}\right\}$ independent

(iii) $\left.\begin{array}{l} X^T A X \sim W_d(r, \Sigma) \\ X^T \underset{\sim}{b} \sim N_d(\underset{\sim}{0}, \|\underset{\sim}{b}\|^2 \Sigma) \end{array}\right\}$ independent

$\iff \forall \underset{\sim}{\ell} \neq \underset{\sim}{0} : \left\{\begin{array}{l} \underset{\sim}{y}^T A \underset{\sim}{y} \sim \sigma_\ell^2 \chi^2(r) \\ \underset{\sim}{y}^T \underset{\sim}{b} \sim N(0, \|\underset{\sim}{b}\|^2 \sigma_\ell^2) \end{array}\right\}$ independent

Proof :

ad (i) Let $X^T A X \sim W_d(r, \Sigma)$ be given

$$\underset{\sim}{y}^T A \underset{\sim}{y} = (X\underset{\sim}{\ell})^T A (X\underset{\sim}{\ell}) = \underset{\sim}{\ell}^T X^T A X \underset{\sim}{\ell} \sim \sigma_\ell^2 \chi^2(r)$$

cf. corollary to th. 2.2

Let $\forall \underset{\sim}{\ell} \neq \underset{\sim}{0} : \underset{\sim}{y}^T A \underset{\sim}{y} \sim \sigma_\ell^2 \chi^2(r)$ be given

$\Rightarrow A^2 = A$ cf. A 6.5

$\Rightarrow \exists \underset{\sim}{a}_1, \dots, \underset{\sim}{a}_r : A = \sum_i \underset{\sim}{a}_i \underset{\sim}{a}_i^T$ where the $\underset{\sim}{a}_i$'s are mutual orthonormal eigenvectors for A , cf. A 6.2

$$\Rightarrow X^TAX = \sum_i X^T \underset{\sim}{a_i} \underset{\sim}{a_i}^T X = \sum_i (X^T \underset{\sim}{a_i})(X^T \underset{\sim}{a_i})^T = \sum_i \underset{\sim}{u_i} \underset{\sim}{u_i}^T$$

$\Rightarrow X^TAX \sim W_d(r, \Sigma)$, as

$$\begin{cases} \underset{\sim}{u_i}\text{'s are independent, cf. lemma 2.3 (iii)} \\ \underset{\sim}{u_i} \sim N_d(\underset{\sim}{0}, \Sigma)\text{, cf. lemma 2.3 (ii) with } \|\underset{\sim}{a_i}\| = 1 \end{cases}$$

ad (ii)

Let $\left.\begin{cases} X^TAX \sim W_d(r, \Sigma) \\ X^TBX \sim W_d(s, \Sigma) \end{cases}\right\}$ independent be given

$$\left.\begin{aligned} y^TAy &= \underset{\sim}{l}^T X^TAX \underset{\sim}{l} \sim \sigma_l^2 \chi^2(r) \\ y^TBy &= \underset{\sim}{l}^T X^TBX \underset{\sim}{l} \sim \sigma_l^2 \chi^2(s) \end{aligned}\right\} \text{cf. corollary to Th. 2.2}$$

y^TAy and y^TBy are independent being functions of independent stochastic matrices

Let $\forall \underset{\sim}{l} : \left.\begin{cases} y^TAy \sim \sigma_l^2 \chi^2(r) \\ y^TBy \sim \sigma_l^2 \chi^2(s) \end{cases}\right\}$ indep. be given

$$\Rightarrow \begin{cases} \left.\begin{aligned} A^2 &= A \\ B^2 &= B \end{aligned}\right\} \text{cf. A 6.5} \\ AB = 0 \quad \text{cf. A 6.6} \end{cases}$$

$$\Rightarrow \begin{cases} \exists\, \underset{\sim}{a_1}, \dots, \underset{\sim}{a_r} : A = \sum_i \underset{\sim}{a_i} \underset{\sim}{a_i}^T \quad 1) \quad \text{cf A 6.2} \\ \exists\, \underset{\sim}{b_1}, \dots, \underset{\sim}{b_s} : B = \sum_j \underset{\sim}{b_j} \underset{\sim}{b_j}^T \quad 2) \quad \text{do} \\ \underset{\sim}{a_i}^T \underset{\sim}{b_j} = (A \underset{\sim}{a_i})^T (B \underset{\sim}{b_j}) = \underset{\sim}{a_i}^T A^T B \underset{\sim}{b_j} \quad 3) \\ \qquad = \underset{\sim}{a_i}^T AB \underset{\sim}{b_j} = 0 \end{cases}$$

1) $\underset{\sim}{a_i}$'s are orthonormal eigenvektors to A

2) $\underset{\sim}{b_j}$'s — — — — B

3) $\underset{\sim}{a_i} = 1 \cdot \underset{\sim}{a_i} = A \underset{\sim}{a_i}$, cf. A 6.1, analogous $\underset{\sim}{b_j} = B \underset{\sim}{b_j}$

$\Rightarrow \quad \underset{\sim}{a}_1, \dots, \underset{\sim}{a}_r, \underset{\sim}{b}_1, \dots, \underset{\sim}{b}_s$ orthonormal

$$\Rightarrow \begin{cases} X^T A X = \sum_i X^T \underset{\sim}{a}_i \underset{\sim}{a}_i^T X = \sum_i (X^T \underset{\sim}{a}_i)(X^T \underset{\sim}{a}_i)^T = \sum_i \underset{\sim}{u}_i \underset{\sim}{u}_i^T \\ X^T B X = \sum_j X^T \underset{\sim}{b}_j \underset{\sim}{b}_j^T X = \sum_j (X^T \underset{\sim}{b}_j)(X^T \underset{\sim}{b}_j)^T = \sum_j \underset{\sim}{v}_j \underset{\sim}{v}_j^T \end{cases}$$

$$\Rightarrow \left.\begin{cases} X^T A X \sim W_d(r, \Sigma) \\ X^T B X \sim W_d(s, \Sigma) \end{cases}\right\} \text{ independent as}$$

$$\begin{cases} \underset{\sim}{u}_i\text{'s and } \underset{\sim}{v}_j\text{'s are all independent, cf. lemma 2.3 (iii)} \\ \underset{\sim}{u}_i, \underset{\sim}{v}_j \sim N_d(\underset{\sim}{0}, \Sigma) \text{ cf. lemma 2.3 (ii) using } \|\underset{\sim}{a}_i\| = 1 \text{ and } \|b_j\| = 1 \end{cases}$$

ad (iii)

$$\text{Let } \left.\begin{cases} X^T A X \sim W_d(r, \Sigma) \\ X^T \underset{\sim}{b} \sim N_d(\underset{\sim}{0}, \|\underset{\sim}{b}\|^2 \Sigma) \end{cases}\right\} \text{ independent be given}$$

$$\left.\begin{aligned} y^T A y &= \underset{\sim}{\ell}^T X^T A X \underset{\sim}{\ell} \sim \sigma_\ell^2 \chi^2(r) \\ y^T b &= \underset{\sim}{b}^T y \sim N(0, \|\underset{\sim}{b}\|^2 \sigma_\ell^2) \end{aligned}\right\} \text{ independent}$$

being functions of independent stoch. variables

$$\text{Let } \forall \underset{\sim}{\ell}: \left.\begin{cases} y^T A y \sim \sigma_\ell^2 \chi^2(r) \\ y^T \underset{\sim}{b} \sim N(0, \|\underset{\sim}{b}\|^2 \sigma_\ell^2) \end{cases}\right\} \text{ indep. be given}$$

$$\Rightarrow \left.\begin{cases} A^2 = A \quad \text{cf. A 6.5} \\ (\underset{\sim}{b}^T y)^2 = y^T \underset{\sim}{b} \underset{\sim}{b}^T y \sim \|\underset{\sim}{b}\|^2 \sigma_\ell^2 \chi^2(1) \end{cases}\right\} \text{ indep.}$$

$$\Rightarrow \begin{cases} \exists \underset{\sim}{a}_1, \dots, \underset{\sim}{a}_r : A = \sum_i \underset{\sim}{a}_i \underset{\sim}{a}_i^T \quad (\underset{\sim}{a}_i\text{'s are orth...}) \\ A\left(\frac{1}{\|\underset{\sim}{b}\|^2} \underset{\sim}{b} \underset{\sim}{b}^T\right) = 0 \Rightarrow A \underset{\sim}{b} \underset{\sim}{b}^T \underset{\sim}{b} = 0 \underset{\sim}{b} \Rightarrow A \underset{\sim}{b} = \underset{\sim}{0} \end{cases}$$

(jf. A 6.6)

$\Rightarrow \quad \underset{\sim}{a}_1, \dots, \underset{\sim}{a}_r, \underset{\sim}{b}$ orthogonal as $\underset{\sim}{a}_i^T \underset{\sim}{b} = (A \underset{\sim}{a}_i)^T \underset{\sim}{b} = \underset{\sim}{a}_i^T A \underset{\sim}{b} = \underset{\sim}{a}_i^T \underset{\sim}{0} = 0$

$\Rightarrow \quad X^T \underset{\sim}{a}_1, \dots, X^T \underset{\sim}{a}_r, X^T \underset{\sim}{b}$ independent cf. lemma 2.3 (iii)

$$\Rightarrow \left.\begin{cases} X^T A X \sim W_d(r, \Sigma) \text{ (cf. (i))} \\ X^T \underset{\sim}{b} \sim N_d(\underset{\sim}{0}, \|\underset{\sim}{b}\|^2 \Sigma) \end{cases}\right\} \text{ independent}$$

(cf lemma 2.3 (ii))

Corollary 1

$$X^T A X \sim W_d(r, \Sigma) \iff A^2 = A$$

Corollary 2

$$\left.\begin{array}{l} X^T A X \sim W_d(r, \Sigma) \\ X^T B X \sim W_d(s, \Sigma) \end{array}\right\} \text{ indep. } \iff \left\{\begin{array}{l} A^2 = A \\ B^2 = B \\ AB = O \end{array}\right.$$

Corollary 3

$$\left.\begin{array}{l} X^T A X \sim W_d(r, \Sigma) \\ X^T \underset{\sim}{b} \sim N_d(\underset{\sim}{0}, \|\underset{\sim}{b}\|^2 \Sigma) \end{array}\right\} \text{ indep. } \iff \left\{\begin{array}{l} A^2 = A \\ A\underset{\sim}{b} = \underset{\sim}{0} \end{array}\right.$$

Non central Wishart distribution

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}_i, \Sigma)$, $i = 1, \dots, m$, independent

$$W = X^T X = \sum_{i=1}^m \underset{\sim}{x}_i \, x_i^T \sim W_d(m, \Sigma; \Delta)$$

$$\Delta = \Sigma^{-\frac{1}{2}} M^T M \Sigma^{-\frac{1}{2}}, \quad M = [\underset{\sim}{\mu}_1 \dots \underset{\sim}{\mu}_m]^T$$

$\uparrow$ non centrality parameter, cf. non central χ^2 dist.

Modified corollaries to theorem 2.4 are valid

Corollary 1 modified

$$X^T A X \sim W_d(r, \Sigma; \Delta_A) \iff A^2 = A$$

Corollary 2 modified

$$\left.\begin{array}{l} X^T A X \sim W_d(r, \Sigma; \Delta_A) \\ X^T B X \sim W_d(s, \Sigma; \Delta_B) \end{array}\right\} \text{ indep. } \iff \left\{\begin{array}{l} A^2 = A \\ B^2 = B \\ AB = O \end{array}\right.$$

Corollary 3 modified

$$\left.\begin{array}{l} X^T A X \sim W(r, \Sigma; \Delta_A) \\ X^T \underset{\sim}{b} \sim N_d(M^T \underset{\sim}{b}, \|\underset{\sim}{b}\|^2 \Sigma) \end{array}\right\} \text{ indep. } \iff \left\{\begin{array}{l} A^2 = A \\ A\underset{\sim}{b} = \underset{\sim}{0} \end{array}\right.$$

Note that

$E[X^T A X] = r\Sigma + M^T A M$ cf. corollary to th. 1.9