Hotellings T^2

$$\underset{\sim}{x} \sim N_d(\underset{\sim}{\mu}, \Sigma), \quad W \sim W_d(m, \Sigma) \quad \text{independent}$$

def: $$T^2 = m(\underset{\sim}{x} - \underset{\sim}{\mu})^T W^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \sim T^2(d, m)$$

$$T^2(d,m) = \frac{md}{m-d+1} F(d, m-d+1) \quad \text{cf. th. 2.8}$$

compare with $t \sim t(m) \Rightarrow t^2 \sim F(1, m)$

Lemma 2.5

$$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix} \sim N_d(\underset{\sim}{0}, \Sigma) \quad ; \quad \underset{\sim}{u}^{(1)} = \begin{bmatrix} u_1 \\ \vdots \\ u_{d-1} \end{bmatrix}$$

$$u_d \mid \underset{\sim}{u}^{(1)} \sim N\left(\beta_0 + \underset{\sim}{\beta}^T \underset{\sim}{u}^{(1)}, \frac{1}{\sigma^{dd}}\right), \quad \text{where}$$

$$\sigma^{jk} = (\Sigma^{-1})_{jk}$$

Proof:

Partition of Σ:

$$\begin{bmatrix} \Sigma_{11} & \underset{\sim}{\sigma}_{1d} \\ \underset{\sim}{\sigma}_{d1}^T & \sigma_{dd} \end{bmatrix}$$

From th 2.1(viii) follows

$$u_d \mid \underset{\sim}{u}^{(1)} \sim N\left(\theta_d + \underset{\sim}{\sigma}_{1d}^T \Sigma_{11}^{-1} (\underset{\sim}{u}^{(1)} - \underset{\sim}{\theta}^{(1)}), \cdot\right)$$

$$= N\left(\theta_d - \underset{\sim}{\sigma}_{1d}^T \Sigma_{11}^{-1} \underset{\sim}{\theta}^{(1)} + \underset{\sim}{\sigma}_{1d}^T \Sigma_{11}^{-1} \underset{\sim}{u}^{(1)}, \cdot\right)$$

$$= N\left(\beta_0 + \underset{\sim}{\beta}^T \underset{\sim}{u}^{(1)}, \cdot\right)$$

$$\mathrm{Var}(u_d \mid \underset{\sim}{u}^{(1)}) = \sigma_{dd} - \underset{\sim}{\sigma}_{d1}^T \Sigma_{11}^{-1} \underset{\sim}{\sigma}_{1d}$$

Note that $\det \Sigma = \det \Sigma_{11} \det(\sigma_{dd} - \underset{\sim}{\sigma}_{d1}^T \Sigma_{11}^{-1} \underset{\sim}{\sigma}_{1d})$ cf. A 3.2

$$= \det \Sigma_{11} (\sigma_{dd} - \underset{\sim}{\sigma}_{d1}^T \Sigma_{11}^{-1} \underset{\sim}{\sigma}_{1d})$$

$$\Rightarrow \operatorname{Var}(u_d \mid \underset{\sim}{u}^{(1)}) = \frac{\det \Sigma}{\det \Sigma_{11}}$$

Note also that

$$(\Sigma^{-1})_{dd} = \left(\frac{\operatorname{adj} \Sigma}{\det \Sigma}\right)_{dd} = \frac{1}{\det \Sigma}(-1)^{d+d} \det \Sigma_{11} = \frac{\det \Sigma_{11}}{\det \Sigma}$$

Hence $\dfrac{\det \Sigma}{\det \Sigma_{11}} = \dfrac{1}{\sigma^{dd}} \Rightarrow \operatorname{Var}(u_d \mid \underset{\sim}{u}^{(1)}) = \dfrac{1}{\sigma^{dd}}$

Lemma 2.6

Linear normal model $y = K\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$ with

K $m \times p$, rank $K = p$ and $\underset{\sim}{\varepsilon} \sim N_m(\underset{\sim}{0}, \sigma^2 I_m)$

$Q = y^T(I_m - P)y \sim \sigma^2 \chi^2(m-p)$, $P = K(K^TK)^{-1}K^T$ projection matrix

$Q = \dfrac{1}{w^{dd}}$, where $W = \begin{bmatrix} K^T \\ y^T \end{bmatrix} [K \;\; y]$

Proof:

$$W = \begin{bmatrix} K^TK & K^Ty \\ y^TK & y^Ty \end{bmatrix} = \begin{bmatrix} W_{11} & \underset{\sim}{w}_{1d} \\ \underset{\sim}{w}_{d1}^T & w_{dd} \end{bmatrix}$$

, W_{11} is not stochastic here

note that $\dfrac{1}{w^{dd}} = \dfrac{\det W}{\det W_{11}}$ cf. lemma 2.5

$$= \frac{\det W_{11} (w_{dd} - \underset{\sim}{w}_{d1}^T W_{11}^{-1} \underset{\sim}{w}_{1d})}{\det W_{11}} \quad \text{cf. A 3.2}$$

$$= y^Ty - y^TK(K^TK)^{-1}K^Ty$$

$$= y^T(I_m - K(K^TK)^{-1}K^T)y$$

$$= y^T(I_m - P)y = Q$$

Lemma 2.7

$W \sim W_d(m, \Sigma) \quad m \geq d$

(i) $\frac{\sigma^{dd}}{w^{dd}} \sim \chi^2(m-d+1)$ indep. of W_{11}

(ii) $\forall \underset{\sim}{\ell} \neq \underset{\sim}{0}: \quad \frac{\underset{\sim}{\ell}^T \Sigma^{-1} \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T W^{-1} \underset{\sim}{\ell}} \sim \chi^2(m-d+1)$

Proof:

ad (i) Let $W = \sum_{i=1}^{m} \underset{\sim}{x}_i \underset{\sim}{x}_i^T$, $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, m$. independent

$= X^T X$

From Lemma 2.5 follows

$x_{id} \mid \underset{\sim}{x}_i^{(1)} \sim N(0 + \underset{\sim}{\beta}^T \underset{\sim}{x}_i^{(1)}, \frac{1}{\sigma^{dd}}) \quad i = 1, \dots, m$

$\Rightarrow \underset{\sim}{x}^{(d)} \mid K \sim N_m(K\underset{\sim}{\beta}, \frac{1}{\sigma^{dd}} I_m), \; K^T = [\underset{\sim}{x}_1^{(1)} \dots \underset{\sim}{x}_m^{(1)}]$

Lemma 2.6 with $\underset{\sim}{y} := \underset{\sim}{x}^{(d)}$ shows

$\frac{1}{w^{dd}} \Big| K \sim \frac{1}{\sigma^{dd}} \chi^2(m-(d-1)) = \frac{1}{\sigma^{dd}} \chi^2(m-d+1)$

The distribution does not depend on the elements of K

$\Rightarrow \frac{1}{w^{dd}} \sim \frac{1}{\sigma^{dd}} \chi^2(m-d+1)$ indep. of K, cf ex. 2.19

$\Rightarrow \frac{\sigma^{dd}}{w^{dd}} \sim \chi^2(m-d+1)$ indep. of $W_{11} = K^T K$

ad (ii) Take an orthogonal matrix L with last row $\frac{1}{\|\underset{\sim}{\ell}\|} \underset{\sim}{\ell}^T$:

$$\frac{\underset{\sim}{\ell}^T \Sigma^{-1} \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T W^{-1} \underset{\sim}{\ell}} = \frac{(\frac{1}{\|\underset{\sim}{\ell}\|}\underset{\sim}{\ell}^T) \Sigma^{-1} (\frac{1}{\|\underset{\sim}{\ell}\|}\underset{\sim}{\ell})}{(\frac{1}{\|\underset{\sim}{\ell}\|}\underset{\sim}{\ell}^T) W^{-1} (\frac{1}{\|\underset{\sim}{\ell}\|}\underset{\sim}{\ell})} = \frac{(L \Sigma^{-1} L^T)_{dd}}{(L W^{-1} L^T)_{dd}}$$

$$= \frac{((L\Sigma L^T)^{-1})_{dd}}{((LWL^T)^{-1})_{dd}} = \frac{(L\Sigma L^T)^{dd}}{(LWL^T)^{dd}} \sim \chi^2(m-d+1)$$

cf. th. 2.2 and lemma 2.7 (i)

(we have used that $L^T = L^{-1}$)

Theorem 2.8 $\quad \underset{\sim}{y} \sim N_d(\underset{\sim}{0}, \Sigma)$, $W \sim W_d(m, \Sigma)$ independent

$$T^2 = m\, \underset{\sim}{y}^T W^{-1} \underset{\sim}{y} \sim \frac{md}{m-d+1} F(d, m-d+1)$$

$$\left(= T^2(d, m) \right)$$

Proof :

$$\frac{T^2}{m} = \underset{\sim}{y}^T W^{-1} \underset{\sim}{y} = \frac{\underset{\sim}{y}^T \Sigma^{-1} \underset{\sim}{y}}{\dfrac{\underset{\sim}{y}^T \Sigma^{-1} \underset{\sim}{y}}{\underset{\sim}{y}^T W^{-1} \underset{\sim}{y}}} = \frac{G}{H}.$$

$H \mid \underset{\sim}{y} \sim \chi^2(m-d+1)$ cf. lemma 2.7 (ii)

the distribution does not depend on the elements of $\underset{\sim}{y}$

$\Rightarrow$ $H \sim \chi^2(m-d+1)$ indep. of $\underset{\sim}{y}$

$G \sim \chi^2(d)$ cf. th. 2.1 (vi) indep. of H

$$\frac{\frac{G}{d}}{\frac{H}{m-d+1}} = \frac{m-d+1}{d} \frac{T^2}{m} \sim F(d, m-d+1)$$

$$\Leftrightarrow \; T^2 \sim \frac{md}{m-d+1} F(d, m-d+1) \Leftrightarrow T^2 \sim T^2(d, m)$$

Corollary $\quad \underset{\sim}{x} \sim N_d(\underset{\sim}{\mu}, \frac{1}{n}\Sigma)$, $W \sim W_d(m, \Sigma)$ indep.

$$T^2 = n\, m\, (\underset{\sim}{x} - \underset{\sim}{\mu})^T W^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \sim T^2(d, m)$$

Proof : $\frac{T^2}{m} = n(\underset{\sim}{x} - \underset{\sim}{\mu})^T W^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) = \underset{\sim}{y}^T W^{-1} \underset{\sim}{y}$ where

$\underset{\sim}{y} = \sqrt{n}(\underset{\sim}{x} - \underset{\sim}{\mu}) \sim N_d(\underset{\sim}{0}, \Sigma) \Rightarrow T^2 \sim T^2(d, m)$

Beta distribution

$$h \sim \chi^2(m_h) \quad , \quad e \sim \chi^2(m_e) \qquad \text{independent}$$

$$t = \frac{h}{e} \sim \frac{m_h}{m_e} F(m_h, m_e)$$
type II beta distribution or invers beta dist.

$$f(t) = \frac{1}{B\left(\frac{m_h}{2}, \frac{m_e}{2}\right)} \frac{t^{\frac{m_h}{2}-1}}{(1+t)^{\frac{m_h+m_e}{2}}} \quad , \quad 0 < t < \infty$$

$$v = \frac{h}{e+h} = \frac{t}{1+t} \sim B\left(\frac{m_h}{2}, \frac{m_e}{2}\right)$$
(type I) beta dist.

$$g(v) = \frac{1}{B\left(\frac{m_h}{2}, \frac{m_e}{2}\right)} v^{\frac{m_h}{2}-1}(1-v)^{\frac{m_e}{2}-1} \quad , \quad 0 < v < 1$$

note that $1 - v = \frac{e}{e+h} = \frac{1}{1+t} = (1+t)^{-1}$

$$B\left(\frac{m_h}{2}, \frac{m_e}{2}\right) \qquad B\left(\frac{m_e}{2}, \frac{m_h}{2}\right) \qquad \frac{m_h}{m_e} F(m_h, m_e)$$

Multi dimensional beta distributions

$$H \sim W_d(m_H, \Sigma) \quad , \quad m_H \geq d$$
$$E \sim W_d(m_E, \Sigma) \quad , \quad m_E \geq d$$
independent

$$T = E^{-\frac{1}{2}} H E^{-\frac{1}{2}} \quad , \quad T \text{ sym.}$$

$$V = (E+H)^{-\frac{1}{2}} H (E+H)^{-\frac{1}{2}} \quad , \quad V \text{ sym.}$$

notation : $g(\underbrace{v_{11}, \dots, v_{dd}}_{\frac{d(d+1)}{2} \text{ variables}}) = g(V)$

Theorem 2.9 :

$$g(V) = \frac{1}{B_d\left(\frac{m_H}{2}, \frac{m_E}{2}\right)} (\det V)^{\frac{m_H-d-1}{2}} \left(\det(I_d - V)\right)^{\frac{m_E-d-1}{2}} ,$$
$$0 < V < I_d$$

proof:

$$f(H,E) = \frac{1}{c_H} (\det H)^{\frac{m_H-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}H\right) \frac{1}{c_E} (\det E)^{\frac{m_E-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}E\right)$$

$$= \frac{1}{c_H c_E} (\det H)^{\frac{m_H-d-1}{2}} (\det E)^{\frac{m_E-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}(E+H)\right)$$

change of variables:

$$V = (E+H)^{-\frac{1}{2}} H (E+H)^{-\frac{1}{2}} \qquad \frac{\partial(H,E)}{\partial(V,Z)} = (\det Z)^{\frac{d+1}{2}} \quad \text{cf A 9.2}$$

$$Z = E+H$$

$$h(V,Z) = \frac{1}{c_H c_E} \left(\det\left(Z^{\frac{1}{2}} V Z^{\frac{1}{2}}\right)\right)^{\frac{m_H-d-1}{2}} \left(\det\left(Z - Z^{\frac{1}{2}} V Z^{\frac{1}{2}}\right)\right)^{\frac{m_E-d-1}{2}} \cdot$$

$$\cdot \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}Z\right) (\det Z)^{\frac{d+1}{2}}$$

$$= \frac{1}{c_H c_E} (\det V)^{\frac{m_H-d-1}{2}} \left(\det(I_d - V)\right)^{\frac{m_E-d-1}{2}} (\det Z)^{\frac{m_H+m_E-d-1}{2}} \cdot$$

$$\cdot \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}Z\right)$$

$$g(V) = \int_{R^{\frac{d(d+1)}{2}}} h(V,Z) \, d\Omega_Z$$

$$= \frac{c_{H+E}}{c_H c_E} (\det V)^{\frac{m_H-d-1}{2}} \left(\det(I_d - V)\right)^{\frac{m_E-d-1}{2}} \cdot$$

$$\int_{R^{\frac{d(d+1)}{2}}} \frac{1}{c_{H+E}} (\det Z)^{\frac{m_H+m_E-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}Z\right) d\Omega_Z$$

$$= \frac{2^{\frac{(m_H+m_E)d}{2}} (\det\Sigma)^{\frac{m_H+m_E}{2}} \Gamma_d\left(\frac{m_H+m_E}{2}\right)}{2^{\frac{m_H d}{2}} (\det\Sigma)^{\frac{m_H}{2}} \Gamma_d\left(\frac{m_H}{2}\right) 2^{\frac{m_E d}{2}} (\det\Sigma)^{\frac{m_E}{2}} \Gamma_d\left(\frac{m_E}{2}\right)} (\det V)^{\frac{m_H-d-1}{2}}$$

$$\left(\det(I_d - V)\right)^{\frac{m_E-d-1}{2}} \cdot 1$$

$$= \frac{1}{B_d\left(\frac{m_H}{2}, \frac{m_E}{2}\right)} (\det V)^{\frac{m_H-d-1}{2}} \left(\det(I_d - V)\right)^{\frac{m_E-d-1}{2}}$$

$$\left.\begin{array}{l} V > 0 \text{ a.s. cf A 5.7} \\ I_d - V > 0 \text{ a.s. due to sym.} \end{array}\right\} \Rightarrow 0 < V < I_d \text{ a.s.}$$

$f(T)$ depends on Σ

$T > 0$

Eigenvalues corresponding to V (and $I_d - V$)

$V > 0$ a.s $\Rightarrow$ eigenvalue $\theta > 0$ a.s cf. A 5.1

$I_d - V > 0$ a.s $\Rightarrow$ $1 - \theta > 0$ a.s. cf. A 1.2 (c) and A 5.1

hence $0 < \theta < 1$ a.s.

$$\det(V - \theta I_d) = \det(E+H)^{-1} \det(H - \theta(E+H))$$
$$= \det(E+H)^{-1} \det((1-\theta)H - \theta E)$$
$$= \det(E+H)^{-1} \det((1-\theta)E) \det\left(HE^{-1} - \frac{\theta}{1-\theta} I_d\right)$$
$$= k \det(HE^{-1} - \varphi I_d), \quad \varphi = \frac{\theta}{1-\theta}$$

$\varphi_1, \dots, \varphi_d$ are different with probability 1 (without proof)

$\Rightarrow \theta_1, \dots, \theta_d$ - - - - - - -

note that $\det V = \prod_j \theta_j$ and $\det(I_d - V) = \prod_j (1-\theta_j)$,
cf. A 1.2 (b) cf. A 1.2 (c)

θ_{max} is often used as test statistic, critical values are tabled

Trace variables

Lawley - Hotelling:

$$T_g^2 = m_E \operatorname{tr}(HE^{-1}) = m_E \sum_{j=1}^{s} \varphi_j = m_E \sum_{j=1}^{s} \frac{\theta_j}{1-\theta_j}$$
$$= m_E U^{(s)}, \quad m_E \geq d, \quad s = \min(d, m_H)$$

Pillai

$$V^{(s)} = \operatorname{tr}(H(E+H)^{-1}) = \sum_{j=1}^{s} \theta_j, \quad m_E \geq d, \quad s = \min(d, m_H)$$

U distribution (Λ distribution)

$$U = \det(I_d - V) = \det(E+H)^{-1} \det E = \frac{\det E}{\det(E+H)}$$

also $U = \prod_{j=1}^{s} (1-\theta_j)$, $s = \min(d, m_H)$

Def : $U \sim U(d, m_H, m_E)$, $m_E \geq d$

$U(d, m_H, m_E) = U(m_H, d, m_E + m_H - d)$ cf. p. 37 top

in particular

$$\frac{1 - U(1, m_H, m_E)}{U(1, m_H, m_E)} = \frac{m_H}{m_E} F(m_H, m_E)$$

$$\frac{1 - U(d, 1, m_E)}{U(d, 1, m_E)} = \frac{d}{m_E + 1 - d} F(d, m_E + 1 - d)$$

$$\frac{1 - (U(2, m_H, m_E))^{\frac{1}{2}}}{(U(2, m_H, m_E))^{\frac{1}{2}}} = \frac{m_H}{m_E - 1} F(2m_H, 2(m_E - 1))$$

$$\frac{1 - (U(d, 2, m_E))^{\frac{1}{2}}}{(U(d, 2, m_E))^{\frac{1}{2}}} = \frac{d}{m_E + 1 - d} F(2d, m_E + 1 - d)$$

When $m_H = 1$ we have

$\operatorname{rank}(H(E+H)^{-1}) = \operatorname{rank} H = 1$

$\Rightarrow \det(H - \theta(E+H)) = 0$ has one root θ

$$U^{(1)} = \frac{\theta}{1-\theta}, \qquad V^{(1)} = \theta$$

$$U = 1-\theta$$

$$T_g^2 = m_E U^{(1)} = m_E \operatorname{tr}(HE^{-1}), \qquad \text{let } H = \underset{\sim}{x}\underset{\sim}{x}^T$$

$\underset{\sim}{x} \sim N_d(\underset{\sim}{0}, \Sigma)$ indep. of E

$$= m_E \operatorname{tr}(\underset{\sim}{x}\underset{\sim}{x}^T E^{-1})$$

$$= m_E \operatorname{tr}(\underset{\sim}{x}^T E^{-1}\underset{\sim}{x}) = m_E \underset{\sim}{x}^T E^{-1}\underset{\sim}{x} \sim T^2(d, m_E)$$

$$m_E U^{(1)} \sim T^2(d, m_E) \iff U^{(1)} \sim \frac{d}{m_E - d + 1} F(d, m_E - d + 1)$$

alternative calculation:

$$U = \frac{\det E}{\det(E+H)} = \frac{\det E}{\det(E + \underset{\sim}{x}\underset{\sim}{x}^T)} = \left(\det\left(I_d + E^{-1}\underset{\sim}{x}\underset{\sim}{x}^T\right)\right)^{-1}$$

$$= \left(\det\left(I_1 + \underset{\sim}{x}^T E^{-1}\underset{\sim}{x}\right)\right)^{-1}$$ cf. hint to exercise 2.12 a (problems 4)

$$= (1 + \underset{\sim}{x}^T E^{-1}\underset{\sim}{x})^{-1} = \left(1 + \frac{T^2}{m_E}\right)^{-1}, \quad T^2 \sim T^2(d, m_E)$$

$$\Rightarrow \quad T^2 = m_E\left(\frac{1}{U} - 1\right) = m_E \frac{1-U}{U} = m_E \frac{\theta}{1-\theta} = m_E U^{(1)} = T_g^2$$

$$\left(U^{(1)}\right)^{-1} \sim \frac{m_E - d + 1}{d} F(m_E - d + 1, d) \Rightarrow$$

$$V^{(1)} = \frac{U^{(1)}}{U^{(1)} + 1} = \left(1 + \left(U^{(1)}\right)^{-1}\right)^{-1} \sim B\left(\frac{d}{2}, \frac{m_E - d + 1}{2}\right)$$

also $$V^{(1)} = \frac{T_g^2}{m_E + T_g^2}$$