Factoriztion of U in beta distributed variables

$$U = \frac{\det E}{\det(E+H)} = b_1 b_2 \dots b_d \,, \text{ where}$$

$$b_k \sim B\left(\frac{m_E - k + 1}{2}, \frac{m_H}{2}\right), \quad k = 1, \dots, d \quad \text{independent}$$

Derivation:

$\det E_k$: k'th leading minor determinant belonging to E, $k = 1, \dots, d$

note that $E_1 = e_{11}$ and $E_d = E$

$$E_k = \begin{bmatrix} E_{k-1} & \underset{\sim}{e}_{k-1} \\ \underset{\sim}{e}_{k-1}^T & e_{kk} \end{bmatrix}$$

$$\Rightarrow \det E_k = \det E_{k-1} \left(e_{kk} - \underset{\sim}{e}_{k-1}^T E_{k-1}^{-1} \underset{\sim}{e}_{k-1} \right)$$

$$= \det E_{k-1} \, \tilde{e}_{kk}$$

$$\Rightarrow \frac{\det E_k}{\det E_{k-1}} = \tilde{e}_{kk}$$

$$\det E = \det E_1 \frac{\det E_2}{\det E_1} \cdots \frac{\det E_d}{\det E_{d-1}} = e_{11} \tilde{e}_{22} \cdots \tilde{e}_{dd}$$

analogous $\det(E+H) = e_{H11} \tilde{e}_{H22} \cdots \tilde{e}_{Hdd}$

$$U = \frac{e_{11} \tilde{e}_{22} \cdots \tilde{e}_{dd}}{e_{H11} \tilde{e}_{H22} \cdots \tilde{e}_{Hdd}} = b_1 b_2 \cdots b_d \,, \text{ where } b_k = \frac{\tilde{e}_{kk}}{\tilde{e}_{Hkk}}, \quad k = 2, \dots, d$$

Let $E = \sum_{i=1}^{m_E} \underset{\sim}{u}_i \underset{\sim}{u}_i^T$ and $H = \sum_{i=m_E+1}^{m_E+m_H} \underset{\sim}{u}_i \underset{\sim}{u}_i^T$, where

$\underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, m_E + m_H$ independent

Lemma 2.5 shows $u_{ik} \mid \underset{\sim}{u}_{i,k-1} \sim N\left(\underset{\sim}{\beta}^T \underset{\sim}{u}_{i,k-1}, \frac{1}{\sigma^{kk}}\right)$

Let
$$\begin{bmatrix} \underset{\sim}{u}_{1,k}^T \\ \underset{\sim}{u}_{2,k}^T \\ \vdots \\ \underset{\sim}{u}_{n,k}^T \end{bmatrix} = \begin{bmatrix} \underset{\sim}{u}_{1,k-1}^T & u_{1k} \\ \underset{\sim}{u}_{2,k-1}^T & u_{2k} \\ \vdots & \vdots \\ \underset{\sim}{u}_{n,k-1}^T & u_{nk} \end{bmatrix} = [K \; \underset{\sim}{y}] = \begin{bmatrix} K_1 & \underset{\sim}{y}_1 \\ K_2 & \underset{\sim}{y}_2 \end{bmatrix},$$

where $n = m_E + m_H$, $\underset{\sim}{y}_1$ m_E-dim, $\underset{\sim}{y}_2$ m_H-dim

$$E_k = \sum_{i=1}^{m_E} \underset{\sim}{u}_{i,k} \underset{\sim}{u}_{i,k}^T = \begin{bmatrix} K_1^T \\ \underset{\sim}{y}_1^T \end{bmatrix} [K_1 \; \underset{\sim}{y}_1] = \begin{bmatrix} K_1^T K_1 & K_1^T \underset{\sim}{y}_1 \\ \underset{\sim}{y}_1^T K_1 & \underset{\sim}{y}_1^T \underset{\sim}{y}_1 \end{bmatrix}$$

From lemma 2.6 and 2.7:

$$Q_k = \underset{\sim}{y}_1^T (I_{m_E} - P_{11}) \underset{\sim}{y}_1, \quad P_{11} = K_1 (K_1^T K_1)^{-1} K_1^T$$

$\sim \frac{1}{\sigma^{kk}} \chi^2(m_E - (k-1))$ indep. of K_1 thus also of K

Note that $Q_k = \underset{\sim}{y}_1^T \underset{\sim}{y}_1 - \underset{\sim}{y}_1^T K_1 (K_1^T K_1)^{-1} K_1^T \underset{\sim}{y}_1$

$= e_{kk} - \underset{\sim}{e}_{k-1}^T E_{k-1}^{-1} \underset{\sim}{e}_{k-1} = \tilde{e}_{kk}$, thus

$\tilde{e}_{kk} \sim \frac{1}{\sigma^{kk}} \chi^2(m_E - k + 1)$ indep. of K

Analogous: $\tilde{e}_{Hkk} = Q_{Hk} \sim \frac{1}{\sigma^{kk}} \chi^2(m_E + m_H - k + 1)$
indep. of K

Let $\tilde{e}_{Hkk}^* = \tilde{e}_{Hkk} - \tilde{e}_{kk}$

Lemma: $\tilde{e}_{Hkk}^* \sim \frac{1}{\sigma^{kk}} \chi^2(m_H)$ indep. of $\tilde{e}_{kk}$

Proof: $\tilde{e}_{kk} = \underset{\sim}{y}_1^T (I_{m_E} - P_{11}) \underset{\sim}{y}_1 , \quad P_{11} = K_1 (K_1^T K_1)^{-1} K_1^T$

$= \underset{\sim}{y}^T (I_n - P_1) \underset{\sim}{y} , \quad n = m_E + m_H$

$P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & I_{m_H} \end{bmatrix}$

$\tilde{e}_{Hkk} = \underset{\sim}{y}^T (I_n - P) \underset{\sim}{y} , \quad P = K (K^T K)^{-1} K^T$

$\tilde{e}^*_{Hkk} = \underset{\sim}{y}^T (I_n - P - (I_n - P_1)) \underset{\sim}{y} = \underset{\sim}{y}^T (P_1 - P) \underset{\sim}{y}$

$= (\underset{\sim}{y} - K \underset{\sim}{\beta})^T (P_1 - P)(\underset{\sim}{y} - K\underset{\sim}{\beta})$ as

$(P_1 - P) K = \begin{bmatrix} P_{11} K_1 \\ K_2 \end{bmatrix} - PK = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} - K = K - K = 0$

$(P_1 - P)^T = P_1^T - P^T = P_1 - P$ thus also $K^T (P_1 - P) = 0$

$$P_1 P = \begin{bmatrix} K_1 (K_1^T K_1)^{-1} K_1^T & 0 \\ 0 & I_{m_H} \end{bmatrix} \begin{bmatrix} K_1 (K^T K)^{-1} K_1^T & K_1 (K^T K)^{-1} K_2^T \\ K_2 (K^T K)^{-1} K_1^T & K_2 (K^T K)^{-1} K_2^T \end{bmatrix}$$

$$= \begin{bmatrix} K_1 (K^T K)^{-1} K_1^T & K_1 (K^T K)^{-1} K_2^T \\ K_2 (K^T K)^{-1} K_1^T & K_2 (K^T K)^{-1} K_2^T \end{bmatrix} = P$$

$P P_1 = (P_1^T P^T)^T = (P_1 P)^T = P^T = P$

$(P_1 - P)^2 = P_1^2 - P_1 P - P P_1 + P^2 = P_1 - P - P + P = P_1 - P$

$\text{rank}(P_1 - P) = \text{tr}\, P_1 - \text{tr}\, P = (k - 1 + m_H) - (k-1) = m_H$ a.s.

$\tilde{e}^*_{Hkk} | K \sim \frac{1}{\sigma^{kk}} \chi^2 (m_H)$ cf. A 6.5 ; the distribution does not depend on K, thus $\tilde{e}^*_{Hkk} \sim \frac{1}{\sigma^{kk}} \chi^2(m_H)$ indep. of K

$(I_n - P_1)(P_1 - P) = P_1 - P - P_1^2 + P_1 P = P_1 - P - P_1 + P = 0$

$\tilde{e}_{kk}$ og $\tilde{e}^*_{Hkk}$ are independent, cf. A 6.6

$b_k = \dfrac{\tilde{e}_{kk}}{\tilde{e}_{Hkk}} = \dfrac{\tilde{e}_{kk}}{\tilde{e}_{kk} + \tilde{e}^*_{Hkk}} \sim B\left(\frac{m_E - k + 1}{2} , \frac{m_H}{2} \right)$ cf. def. of beta distr.

indep. of K

b_k indep. of b_{k-1}, as b_{k-1} is a function of K

$\Rightarrow b_k$ indep. of $(b_{k-1}, \ldots, b_1) \Rightarrow b_1, \ldots, b_d$ independent

Cholesky factorization of E and $E+H$ (unique, cf. A S.11)

$$E = C^T C \Rightarrow E_k = C_k^T C_k$$

$$\Rightarrow \det E_k = (\det C_k)^2 = c_{11}^2 c_{22}^2 \dots c_{kk}^2$$

$$\Rightarrow \tilde{e}_{kk} = c_{kk}^2$$

analogous $E + H = D^T D \Rightarrow \dots \Rightarrow \tilde{e}_{Hkk} = d_{kk}^2$

$$b_k = \frac{c_{kk}^2}{d_{kk}^2} \Leftrightarrow \frac{1}{b_k} - 1 = \frac{d_{kk}^2 - c_{kk}^2}{c_{kk}^2}$$

$$\frac{d_{kk}^2 - c_{kk}^2}{c_{kk}^2} \sim \frac{m_H}{m_E - k + 1} F(m_H, m_E - k + 1)$$

cf. formula (2.21)

Specified to $m_H = 1$:

$$U = \left(1 + \frac{T^2}{m_E}\right)^{-1}, \quad T^2 = m_E \underset{\sim}{x}^T E \underset{\sim}{x}, \quad H = \underset{\sim}{x}\underset{\sim}{x}^T$$
$$T^2 \sim T^2(d, m_E)$$

cf. formula (2.58)

$$\Rightarrow \frac{\det E_k}{\det (E+H)_k} = \left(1 + \frac{T_k^2}{m_E}\right)^{-1}, \quad T_k^2 \sim T^2(k, m_E)$$

$$b_k = \frac{\tilde{e}_{kk}}{\tilde{e}_{Hkk}} = \frac{\det E_k}{\det E_{k-1}} \frac{\det (E+H)_{k-1}}{\det (E+H)_k} = \frac{\det E_k}{\det (E+H)_k} \frac{\det (E+H)_{k-1}}{\det E_{k-1}}$$

$$= \frac{1 + \frac{T_{k-1}^2}{m_E}}{1 + \frac{T_k^2}{m_E}} = \frac{m_E + T_{k-1}^2}{m_E + T_k^2} \Rightarrow \frac{1}{b_k} - 1 = \frac{T_k^2 - T_{k-1}^2}{m_E + T_{k-1}^2}$$

$$\frac{T_k^2 - T_{k-1}^2}{m_E + T_{k-1}^2} \sim \frac{1}{m_E - k + 1} F(1, m_E - k + 1)$$

cf. formula (2.21)

Factorization of U in two U distributed variables

$$U = \frac{\det E}{\det(E+H)} \sim U(d, m_H, m_E)$$

Let $E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$ and $E+H = \begin{bmatrix} (E+H)_{11} & (E+H)_{12} \\ (E+H)_{21} & (E+H)_{22} \end{bmatrix}$,

where E_{11} ev $d_1 \times d_1$ and E_{22} $d_2 \times d_2$, analogous $E+H$

We have $U = b_{(1)} b_{(2)}$, where

$$\left.\begin{array}{l} b_{(1)} \sim U(d_1, m_H, m_E) \\ b_{(2)} \sim U(d_2, m_H, m_E - d_1) \end{array}\right\} \text{independent}$$

Derivation:

$$U = \frac{\det E}{\det(E+H)} = \frac{\det E_{11}}{\det(E+H)_{11}} \frac{\det E_{22\cdot 1}}{\det(E+H)_{22\cdot 1}} = b_{(1)} b_{(2)},$$

where $E_{22\cdot 1} = E_{22} - E_{21} E_{11}^{-1} E_{12}$

and $(E+H)_{22\cdot 1} = (E+H)_{22} - (E+H)_{21} (E+H)_{11}^{-1} (E+H)_{12}$

Obviously we have

$$b_{(1)} = \frac{\det E_{11}}{\det(E_{11}+H_{11})} \sim U(d_1, m_H, m_E), \quad \text{as}$$

$$\left.\begin{array}{l} E_{11} \sim W_{d_1}(m_E, \Sigma_{11}) \\ H_{11} \sim W_{d_1}(m_H, \Sigma_{11}) \end{array}\right\} \text{ independent}$$

Let $m_E + m_H = n$

$$E = \sum_{i=1}^{m_E} \underset{\sim}{u}_i \underset{\sim}{u}_i^T, \quad H = \sum_{i=m_E+1}^{n} \underset{\sim}{u}_i \underset{\sim}{u}_i^T, \quad \underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma), \; i = 1, \dots, n \text{ indep.}$$

$$[\underset{\sim}{u}_1 \dots \underset{\sim}{u}_{m_E} \underset{\sim}{u}_{m_E+1} \dots \underset{\sim}{u}_n]^T = \begin{array}{cc} \begin{bmatrix} K_1 & Y_1 \\ K_2 & Y_2 \end{bmatrix} & \begin{array}{c} m_E \\ m_H \end{array} \\ \begin{array}{cc} d_1 & d_2 \end{array} & \end{array}$$

$$= [K \;\; Y]$$

$$H_{22\cdot1}^* = (E+H)_{22\cdot1} - E_{22\cdot1}$$

Lemma : $\left.\begin{array}{l} H_{22\cdot1}^* \sim W_{d_2}(m_H, \Sigma_{22\cdot1}) \\ E_{22\cdot1} \sim W_{d_2}(m_E - d_1, \Sigma_{22\cdot1}) \end{array}\right\}$ independent

Proof : $E = \begin{bmatrix} K_1^T \\ Y_1^T \end{bmatrix} [K_1 \; Y_1] = \begin{bmatrix} K_1^T K_1 & K_1^T Y_1 \\ Y_1^T K_1 & Y_1^T Y_1 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$

$$E_{22\cdot1} = Y_1^T Y_1 - Y_1^T K_1 (K_1^T K_1)^{-1} K_1^T Y_1$$

$$= Y_1^T (I_{m_E} - P_{11}) Y_1, \quad P_{11} = K_1 (K_1^T K)^{-1} K_1^T$$

$$= Y^T (I_n - P_1) Y, \quad P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & I_{m_H} \end{bmatrix}$$

Analogous

$$(E+H)_{22\cdot1} = Y^T (I_n - P) Y, \quad P = K (K^T K)^{-1} K^T$$

$$H^{*}_{22\cdot1} = Y^T(P_1 - P)Y$$

$(P_1-P)|K$ is a projektion matrix, cf. previous lemma

$\text{rank}(P_1-P) = \text{tr}\,P_1 - \text{tr}\,P = (d_1+m_H) - d_1 = m_H$ a.s

$\text{Var}(y_i \mid \underset{\sim}{k}_i) = \Sigma_{22\cdot1}$ cf. th. 2.1 (viii)

$H^{*}_{22\cdot1} \mid K \sim W_{d_2}(m_H, \Sigma_{22\cdot1})$ cf corollary 1 to th. 2.4

The distribution does not depend on K, thus

$H^{*}_{22\cdot1} \sim W_{d_2}(m_H, \Sigma_{22\cdot1})$ indep. of. K

$\text{rank}(I_n - P_1) = n - (d_1 + m_H) = m_E + m_H - d_1 - m_H$
$= m_E - d_1$

$E_{22\cdot1} \mid K_1 \sim W_{d_2}(m_E - d_1, \Sigma_{22\cdot1})$ cf. corollary 1 to th. 2.4

The distribution does not depend on K_1, thus

$E_{22\cdot1} \sim W_{d_2}(m_E - d_1, \Sigma_{22\cdot1})$ independent of K_1 thus also of K

$(P_1-P)(I_n - P_1) = 0$

$H^{*}_{22\cdot1}$ og $E_{22\cdot1}$ indep. cf. corollary 2 to th. 2.4

$$b_{(2)} = \frac{\det E_{22\cdot1}}{\det(E+H)_{22\cdot1}} = \frac{\det E_{22\cdot1}}{\det(E_{22\cdot1} + H^{*}_{22\cdot1})} \sim U(d_2, m_H, m_E - d_1)$$

independent of K thus also of $b_{(1)}$

$m_H = 1$:

$$U = b_{(1)} \, b_{(2)}$$

$$\Rightarrow b_{(2)} = \frac{U}{b_{(1)}} = \frac{\det E}{\det(E+H)} \, \frac{\det(E+H)_{11}}{\det E_{11}}$$

$$= \frac{\det E}{\det E_{11}} \, \frac{\det(E+H)_{11}}{\det(E+H)} = \frac{1 + \frac{T_{d_1}^2}{m_E}}{1 + \frac{T_d^2}{m_E}}$$ cf. formula (2.58)

$$\frac{1}{b_{(2)}} - 1 = \frac{1 - b_{(2)}}{b_{(2)}} = \frac{T_d^2 - T_{d_1}^2}{m_E + T_{d_1}^2} \sim \frac{d_2}{m_E - d + 1} \, F(d_2, m_E - d + 1),$$

cf. formula (2.42) with $d := d_2$ and $m_E := m_E - d_1$

$b_{(2)}$ indep. of $T_{d_1}^2$ ($T_{d_1}^2$ is a function of $b_{(1)}$)

thus $\frac{T_d^2 - T_{d_1}^2}{m_E + T_{d_1}^2}$ is indep. of $T_{d_1}^2$