$W \sim W_d(m, \Sigma)$, $m \geq d$

Let $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ $\quad W_{11}$ $d_1 \times d_1$, $\quad W_{22}$ $d_2 \times d_2$

and $W_{22\cdot 1} = W_{22} - W_{21} W_{11}^{-1} W_{12}$

Lemma 2.10

$$W_{22\cdot 1} \sim W_{d_2}(m - d_1, \Sigma_{22\cdot 1}) \quad \text{indep. of } (W_{11}, W_{12})$$

Proof:

Assume $W = \sum_i \underset{\sim}{x}_i \underset{\sim}{x}_i^T$, $\quad \underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \ldots, m$ independent

Let $\underset{\sim}{x}_i = \begin{bmatrix} \underset{\sim}{u}_i \\ \underset{\sim}{v}_i \end{bmatrix}$, $\underset{\sim}{u}_i$ d_1-dim., $\underset{\sim}{v}_i$ d_2-dim.

$$W = \begin{bmatrix} \sum_i \underset{\sim}{u}_i \underset{\sim}{u}_i^T & \sum_i \underset{\sim}{u}_i \underset{\sim}{v}_i^T \\ \sum_i \underset{\sim}{v}_i \underset{\sim}{u}_i^T & \sum_i \underset{\sim}{v}_i \underset{\sim}{v}_i^T \end{bmatrix} = \begin{bmatrix} U^T U & U^T V \\ V^T U & V^T V \end{bmatrix}$$

$$\begin{aligned} W_{22\cdot 1} &= V^T V - V^T U (U^T U)^{-1} U^T V \\ &= V^T (I_m - U (U^T U)^{-1} U^T) V \\ &= V^T (I_m - P) V \end{aligned}$$

Note that $P|U$ is a projection matrix

$$\operatorname{rank} P = \operatorname{rank} U = d_1 \text{ a.s.}$$

Let $\underset{\sim}{v}_{i\cdot 1} = \underset{\sim}{v}_i - \Sigma_{21} \Sigma_{11}^{-1} \underset{\sim}{u}_i$

$$V_{2\cdot1} = \begin{bmatrix} \underset{\sim}{v}_{1\cdot1}^T \\ \vdots \\ \underset{\sim}{v}_{m\cdot1}^T \end{bmatrix} = V - U\Sigma_{11}^{-1}\Sigma_{12}$$

Note that $(I_m - P)V_{2\cdot1} = (I_m - P)(V - U\Sigma_{11}^{-1}\Sigma_{12})$

$$= (I_m - P)V - (U - U)\Sigma_{11}^{-1}\Sigma_{12}$$

$$= (I_m - P)V$$

$$\Rightarrow \quad W_{22\cdot1} = V_{2\cdot1}^T (I_m - P) V_{2\cdot1}$$

$\underset{\sim}{v}_i \mid \underset{\sim}{u}_i \sim N_{d_2}(\underset{\sim}{0} + \Sigma_{21}\Sigma_{11}^{-1}(\underset{\sim}{u}_i - \underset{\sim}{0}), \Sigma_{22\cdot1})$ cf. th. 2.1 (viii)

$= N_{d_2}(\Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{u}_i, \Sigma_{22\cdot1})$, $i = 1, \ldots, m$

independent

$\Rightarrow \quad \underset{\sim}{v}_{i\cdot1} \mid \underset{\sim}{u}_i \sim N_{d_2}(\underset{\sim}{0}, \Sigma_{22\cdot1})$, $i = 1, \ldots, m$

independent

$\Rightarrow \quad V_{2\cdot1}^T (I_m - P) V_{2\cdot1} \mid U \sim W_{d_2}(m - d_1, \Sigma_{22\cdot1})$

cf. th 2.4 corollary 1

The distribution does not depend on U

$\Rightarrow \quad V_{2\cdot1}^T (I_m - P) V_{2\cdot1} \sim W_{d_2}(m - d_1, \Sigma_{22\cdot1})$

Note that $P(I_m - P)^T = P(I_m - P) = P - P = 0$

$$\Rightarrow \left\{ \begin{array}{l} P V_{2\cdot1} I_{d_2} \mid U \\ \\ (I_m - P) V_{2\cdot1} I_{d_2} \mid U \end{array} \right\}$$ independent (cf. ex. 2.18)

$\Rightarrow \quad P V_{2\cdot1} \mid U$ and $W_{22\cdot1} \mid U$ independent

$\Rightarrow \quad (P V_{2\cdot1}, U)$ and $W_{22\cdot1}$ independent, as $x|u$ and $y|u$ indep. $\wedge$ y and u indep.

$$\Rightarrow \quad f(x,y|u) = f(x|u)\, f(y|u) = f(x|u)\, f(y)$$

$$\Rightarrow \quad f(x,y,u) = f(x,u)\, f(y)$$

$\Rightarrow$ (x,u) and y independent

Note that $W_{11} = U^T U$

$$W_{12} = U^T V = U^T P V = U^T P (V_{2.1} + U \Sigma_{11}^{-1} \Sigma_{12})$$
$$= U^T P V_{2.1} + U^T U \Sigma_{11}^{-1} \Sigma_{12}$$

that is W_{11} and W_{12} are both functions of $(PV_{2.1}, U)$

thus $W_{22.1}$ and (W_{11}, W_{12}) are independent

Note that $W_{21} W_{11}^{-1} W_{12}$ is independent of $W_{22.1}$, as $W_{21} W_{11}^{-1} W_{12}$ is a function of (W_{11}, W_{12}).

Corollary:

$$\Sigma_{12} = 0 \Rightarrow W_{21} W_{11}^{-1} W_{12} \sim W_{d_2}(d_1, \Sigma_{22})$$

Proof:

$$\Sigma_{12} = 0 \Rightarrow \Sigma_{22.1} = \Sigma_{22} \wedge V_{2.1} = V$$

Note that
$$W_{21} W_{11}^{-1} W_{12} = W_{22} - W_{22.1}$$
$$= V^T V - V^T (I_{n} - P) V$$
$$= V^T P V$$
$$= V_{2.1}^T P V_{2.1}$$

$$\Rightarrow \quad W_{21} W_{11}^{-1} W_{12} \mid U \sim W_{d_2}(d_1, \Sigma_{22})$$

$$\Rightarrow \quad W_{21} W_{11}^{-1} W_{12} \sim W_{d_2}(d_1, \Sigma_{22})$$ indep. of U thus indep. of W_{11}

$$\left.\begin{array}{l} \underset{\sim}{x} \sim N_d(\underset{\sim}{\mu}, \Sigma) \\ W \sim W_d(m, \Sigma) \end{array}\right\} \text{ independent}$$

Let $\underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{x}^{(1)} \\ \underset{\sim}{x}^{(2)} \end{bmatrix}$ and $\underset{\sim}{\mu} = \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \end{bmatrix}$ $\quad \underset{\sim}{x}^{(1)}, \underset{\sim}{\mu}^{(1)}$ d_1-dim. $\quad \underset{\sim}{x}^{(2)}, \underset{\sim}{\mu}^{(2)}$ d_2-dim.

$$\left.\begin{array}{l} \Delta_d^2 = \underset{\sim}{\mu}^T \Sigma^{-1} \underset{\sim}{\mu} \\ \Delta_{d_1}^2 = \underset{\sim}{\mu}^{(1)T} \Sigma_{11}^{-1} \underset{\sim}{\mu}^{(1)} \end{array}\right\} \text{ Mahalanobis' squared distances}$$

$$\Delta_d^2 - \Delta_{d_1}^2 = \underset{\sim}{\mu}_{2\cdot1}^T \Sigma_{22\cdot1}^{-1} \underset{\sim}{\mu}_{2\cdot1}, \quad \underset{\sim}{\mu}_{2\cdot1} = \underset{\sim}{\mu}^{(2)} - \Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{\mu}^{(1)}$$

cf. ex. 2.20

Note that $\Delta_d^2 = \Delta_{d_1}^2 \iff \underset{\sim}{\mu}_{2\cdot1} = \underset{\sim}{0}$ $\quad (\Sigma_{22\cdot1} > 0)$

Sample versions:

$$T_d^2 = m\, \underset{\sim}{x}^T W^{-1} \underset{\sim}{x}$$

$$T_{d_1}^2 = m\, \underset{\sim}{x}^{(1)T} W_{11}^{-1} \underset{\sim}{x}^{(1)}$$

$$T_d^2 - T_{d_1}^2 = m\, \underset{\sim}{x}_{2\cdot1}^T W_{22\cdot1}^{-1} \underset{\sim}{x}_{2\cdot1}, \quad \underset{\sim}{x}_{2\cdot1} = \underset{\sim}{x}^{(2)} - W_{21}W_{11}^{-1}\underset{\sim}{x}^{(1)}$$

cf. ex. 2.20

Theorem 2.11

When $H_0 : \underset{\sim}{\mu}_{2\cdot1} = \underset{\sim}{0}$ is true we have

$$\frac{T_d^2 - T_{d_1}^2}{m + T_{d_1}^2} \sim \frac{d_2}{m-d+1} F(d_2, m-d+1)$$

indep. of $T_{d_1}^2$ (only when H_0 is true)

Proof: Let $W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} U^T U & U^T V \\ V^T U & V^T V \end{bmatrix} \begin{matrix} d_1 \\ d_2 \end{matrix}$

$$\underset{\sim}{x}^{(2)} \mid \underset{\sim}{x}^{(1)} \sim N_{d_2}\left(\underset{\sim}{\mu}^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}), \Sigma_{22\cdot 1} \right)$$
$$= N_{d_2}\left(\Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{x}^{(1)}, \Sigma_{22\cdot 1} \right), \ H_0 \text{ true}$$

$$\underset{\sim}{x}_{2\cdot 1} = \underset{\sim}{x}^{(2)} - W_{21}W_{11}^{-1}\underset{\sim}{x}^{(1)}$$

Note that $W_{21}W_{11}^{-1}\underset{\sim}{x}^{(1)} = V^T U W_{11}^{-1}\underset{\sim}{x}^{(1)} = V^T \underset{\sim}{a}$
$$= a_1 \underset{\sim}{v}_1 + \dots + a_m \underset{\sim}{v}_m$$

$\underset{\sim}{x}_{2\cdot 1} \mid (\underset{\sim}{x}^{(1)}, U) \sim N_{d_2}$, as $\underset{\sim}{x}_{2\cdot 1}$ is a lin. comb. of $m+1$ indep. d_2-dim. normal distributed vectors ($\underset{\sim}{x}^{(2)}, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_m$ independent)

Note that $E\left[W_{21}W_{11}^{-1}\underset{\sim}{x}^{(1)} \mid (\underset{\sim}{x}^{(1)}, U) \right]$
$$= E\left[V^T U W_{11}^{-1}\underset{\sim}{x}^{(1)} \mid (\underset{\sim}{x}^{(1)}, U) \right]$$
$$= E\left[V^T \mid (\underset{\sim}{x}^{(1)}, U) \right] U W_{11}^{-1}\underset{\sim}{x}^{(1)}$$
$$= \Sigma_{21}\Sigma_{11}^{-1} U^T U W_{11}^{-1}\underset{\sim}{x}^{(1)}, \text{ as } E[\underset{\sim}{v}_i \mid \underset{\sim}{u}_i] = \Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{u}_i$$
cf. page 2
$$= \Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{x}^{(1)}$$

$$E\left[\underset{\sim}{x}_{2\cdot 1} \mid (\underset{\sim}{x}^{(1)}, U) \right] = E\left[\underset{\sim}{x}^{(2)} \mid \underset{\sim}{x}^{(1)} \right] - E\left[W_{21}W_{11}^{-1}\underset{\sim}{x}^{(1)} \mid (\underset{\sim}{x}^{(1)}, U) \right]$$
$$= \underset{\sim}{\mu}^{(2)} + \Sigma_{21}\Sigma_{11}^{-1}(\underset{\sim}{x}^{(1)} - \underset{\sim}{\mu}^{(1)}) - \Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{x}^{(1)}$$
$$= \underset{\sim}{\mu}^{(2)} - \Sigma_{21}\Sigma_{11}^{-1}\underset{\sim}{\mu}^{(1)} = \underset{\sim}{\mu}_{2\cdot 1} \ (= \underset{\sim}{0}, H_0 \text{ true})$$

Note that $Var(\underset{\sim}{v}_i \mid U) = Var(\underset{\sim}{v}_i \mid \underset{\sim}{u}_i) = \Sigma_{22\cdot 1}$

$$Var\left(\underset{\sim}{x}_{2\cdot 1} \mid (\underset{\sim}{x}^{(1)}, U) \right) = \Sigma_{22\cdot 1} + \underbrace{\sum_i a_i^2 \Sigma_{22\cdot 1}}_{\text{cf. ex. 2.6}} = \left(1 + \sum_i a_i^2\right)\Sigma_{22\cdot 1}$$

$$\underset{\sim}{x}_{2\cdot 1} \,\Big|\, (\underset{\sim}{x}^{(1)}, U) \sim N_{d_2}\left(\underset{\sim}{\mu}_{2\cdot 1}, \left(1+\sum_i a_i^2\right) \Sigma_{22\cdot 1}\right)$$

Let $y = \dfrac{\underset{\sim}{x}_{2\cdot 1}}{\sqrt{1+\sum_i a_i^2}}$

When H_0 is true, $y \,\Big|\, (\underset{\sim}{x}^{(1)}, U) \sim N_{d_2}(\underset{\sim}{0}, \Sigma_{22\cdot 1})$

The distribution does neither depend on $(\underset{\sim}{x}^{(1)}, U)$ nor on $\sum_i a_i^2$, as $\underset{\sim}{a} = U W_{11}^{-1} \underset{\sim}{x}^{(1)}$

Note that
$$\begin{aligned}\sum_i a_i^2 &= \underset{\sim}{a}^T \underset{\sim}{a} = (U W_{11}^{-1} \underset{\sim}{x}^{(1)})^T (U W_{11}^{-1} \underset{\sim}{x}^{(1)}) \\ &= \underset{\sim}{x}^{(1)T} W_{11}^{-1} U^T U W_{11}^{-1} \underset{\sim}{x}^{(1)} \\ &= \underset{\sim}{x}^{(1)T} W_{11}^{-1} \underset{\sim}{x}^{(1)} = \frac{T_{d_1}^2}{m}\end{aligned}$$

thus y is independent of $T_{d_1}^2$

Moreover $W_{22\cdot 1}$ is independent of $(T_{d_1}^2, y)$, as $(T_{d_1}^2, y)$ is a function of $(W_{11}, W_{12}, \underset{\sim}{x})$, cf. lemma 2.10

$$\left.\begin{array}{l} W_{22\cdot 1} \text{ and } (T_{d_1}^2, y) \text{ indep.} \\ T_{d_1}^2 \text{ and } y \text{ indep.} \end{array}\right\} \Rightarrow W_{22\cdot 1}, T_{d_1}^2, y \text{ indep.}$$

$$(m-d_1)\, y^T W_{22\cdot 1}^{-1} y \sim T^2(d_2, m-d_1) \text{ indep. of } T_{d_1}^2$$

Now
$$(m-d_1)\, y^T W_{22\cdot 1}^{-1} y = \frac{(m-d_1)\, \underset{\sim}{x}_{2\cdot 1}^T W_{22\cdot 1}^{-1} \underset{\sim}{x}_{2\cdot 1}}{1+\sum_i a_i^2} \frac{m}{m}$$

$$= \frac{(m-d_1)(T_d^2 - T_{d_1}^2)}{m + T_{d_1}^2} \sim \frac{(m-d_1)\,d_2}{(m-d_1)-d_2+1} F(d_2, (m-d_1)-d_2+1)$$

independent of $T_{d_1}^2$

$$\Rightarrow \frac{T_d^2 - T_{d_1}^2}{m + T_{d_1}^2} \sim \frac{d_2}{m-d+1} F(d_2, m-d+1)$$

independent of $T_{d_1}^2$

Note the transformation

$$\frac{T_d^2 - T_{d_1}^2}{m + T_{d_1}^2} = \frac{1 + \frac{T_d^2}{m} - \left(1 + \frac{T_{d_1}^2}{m}\right)}{1 + \frac{T_{d_1}^2}{m}} = \frac{1-u}{u},$$

$$\text{where } u = \frac{1 + \frac{T_{d_1}^2}{m}}{1 + \frac{T_d^2}{m}}$$

Skewness and kurtosis for a d-dimensional stochastic vector

Skewness: $\beta_{1,d} = E\left[\left((\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu})\right)^3\right]$, where

$\underset{\sim}{x}, \underset{\sim}{y}$ independent, identical distributed

Kurtosis: $\beta_{2,d} = E\left[\left((\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})\right)^2\right]$

Analogue empirical values $\left(\hat{\Sigma} = \frac{Q}{n}\right)$:

Skewness: $b_{1,d} = \frac{1}{n^2} \sum_i \sum_j \left((\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} (\underset{\sim}{x}_j - \bar{\underset{\sim}{x}})\right)^3$

Kurtosis: $b_{2,d} = \frac{1}{n} \sum_i \left((\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} (\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})\right)^2$