Power of the test

$$T^2 = n(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu_0})^T S^{-1} (\underset{\sim}{\bar{x}} - \underset{\sim}{\mu_0})$$

$$= (n-1)\left(\sqrt{n}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu_0})\right)^T Q^{-1} \left(\sqrt{n}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu_0})\right)$$

$$\sim T^2(d, n-1, \delta) \text{ when } H_1 \text{ is true , where}$$

$$\delta = \left(\sqrt{n}(\underset{\sim}{\mu} - \underset{\sim}{\mu_0})\right)^T \Sigma^{-1} \left(\sqrt{n}(\underset{\sim}{\mu} - \underset{\sim}{\mu_0})\right)$$

$$= n(\underset{\sim}{\mu} - \underset{\sim}{\mu_0})^T \Sigma^{-1} (\underset{\sim}{\mu} - \underset{\sim}{\mu_0})$$

$$\Rightarrow \quad F = \frac{n-d}{(n-1)d} T^2 \sim F(d, n-d, \delta)$$

$$\gamma = P(F \geq F_{1-\alpha}(d, n-d))$$

In tables δ is often "standardized":

$$\varphi = \sqrt{\frac{\delta}{\nu_1 + 1}} = \sqrt{\frac{\delta}{d+1}}$$

Step-down test

$$H_0 : \underset{\sim}{\mu} = \underset{\sim}{0}$$

Let $\underset{\sim}{\mu_{k-1}} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{k-1} \end{bmatrix}$, $k = 2, \dots, d$

$$\underset{\sim}{\gamma}_{k-1}^T = \underset{\sim}{\sigma}_{k,k-1}^T \Sigma_{k-1}^{-1} , \quad \Sigma_k = \begin{array}{cc} \begin{bmatrix} \Sigma_{k-1} & \underset{\sim}{\sigma}_{k,k-1} \\ \underset{\sim}{\sigma}_{k-1,k}^T & \sigma_{kk} \end{bmatrix} & \begin{matrix} k-1 \\ 1 \end{matrix} \end{array}$$

$$H_{0k} : \mu_k - \underset{\sim}{\gamma}_{k-1}^T \underset{\sim}{\mu}_{k-1} = 0 , \quad k = 2, \dots, d , \quad H_{01} : \mu_1 = 0$$

Theorem 2.11 with $\underset{\sim}{\mu}^{(1)} = \underset{\sim}{\mu}_{k-1}$, $\underset{\sim}{\mu}^{(2)} = \mu_k$, $m = n-1$,

$d = k$ and $d_2 = 1$ is used :

$$F_k = \frac{(n-1)-k+1}{1} \frac{T_k^2 - T_{k-1}^2}{(n-1)+T_{k-1}^2} = \frac{(n-k)(T_k^2 - T_{k-1}^2)}{n-1+T_{k-1}^2}$$

$\sim F(1, n-k)$ is test stastistic for H_{0k}, $k = 2, \dots, d$

(note that $F(1, n-k) = t^2(n-k)$)

in particular $F_1 = \dfrac{(n-1)(T_1^2 - 0)}{n-1+0} = T_1^2 \sim F(1, n-1) = t^2(n-1)$

F_k independent of T_{k-1}^2 when H_0 is true, cf. th. 2.11

$\Rightarrow$ F_k independent of $(F_{k-1}, \dots, F_1)$ (without proof)

$\Rightarrow$ $F_k, \dots, F_1$ independent, $k = 2, \dots, d$

The significance level α_k i k'th test has the following connection with the overall significance level α:

$$\alpha = 1 - \prod_{k=1}^{d} (1 - \alpha_k)$$

Linear constraints on the mean

$\underset{\sim}{x}_i \sim N_2(\underset{\sim}{\mu}, \Sigma)$, $\underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$i = 1, \dots, n$ independent

$H_0: \mu_1 - \mu_2 = c$ is the paired t-test

$y_i = x_{i1} - x_{i2} \sim N(\mu_1 - \mu_2, \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) = N(\mu_y, \sigma_y^2)$,

$i = 1, \dots, n$

$$t = \frac{\bar{y} - c}{\frac{s_y}{\sqrt{n}}} \sim t(n-1) \text{ when } H_0 \text{ is true}$$

For test of $H_0: a_1\mu_1 + a_2\mu_2 = c \Leftrightarrow \underset{\sim}{a}^T \underset{\sim}{\mu} = c$

is used a similar test statistic setting

$y_i = a_1 x_{i1} + a_2 x_{i2} \Leftrightarrow y_i = \underset{\sim}{a}^T \underset{\sim}{x}_i$

Generalization:

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$, independent

$H_0: A\underset{\sim}{\mu} = \underset{\sim}{b}$, A $q \times d$, rank $A = q \le d$

$\underset{\sim}{y}_i = A\underset{\sim}{x}_i \sim N_q(A\underset{\sim}{\mu}, A\Sigma A^T)$, cf. th. 2.1 (i),
$i = 1, \dots, n$, independent

$T^2 = n(\bar{\underset{\sim}{y}} - \underset{\sim}{b})^T S_y^{-1}(\bar{\underset{\sim}{y}} - \underset{\sim}{b}) = n(A\bar{\underset{\sim}{x}} - \underset{\sim}{b})^T (ASA^T)^{-1}(A\bar{\underset{\sim}{x}} - \underset{\sim}{b})$

$\sim T^2(q, n-1)$ when H_0 is true

Let $\underset{\sim}{\mu}_1$ be any solution to $A\underset{\sim}{\mu} = \underset{\sim}{b}$ and

$\underset{\sim}{\gamma} = \underset{\sim}{\mu} - \underset{\sim}{\mu}_1$

$\underset{\sim}{z}_i = \underset{\sim}{x}_i - \underset{\sim}{\mu}_1 \sim N_d(\underset{\sim}{\gamma}, \Sigma)$, $i = 1, \dots, n$, independent

Note that $A\underset{\sim}{\mu} = \underset{\sim}{b} \Leftrightarrow A\underset{\sim}{\mu} - \underset{\sim}{b} = \underset{\sim}{0} \Leftrightarrow A\underset{\sim}{\mu} - A\underset{\sim}{\mu}_1 = \underset{\sim}{0}$

$\Leftrightarrow A(\underset{\sim}{\mu} - \underset{\sim}{\mu}_1) = \underset{\sim}{0} \Leftrightarrow A\underset{\sim}{\gamma} = \underset{\sim}{0}$

thus $H_0: A\underset{\sim}{\gamma} = \underset{\sim}{0}$ with

$T_z^2 = n(A\bar{\underset{\sim}{z}})^T (ASA^T)^{-1} A\bar{\underset{\sim}{z}}$, $A\bar{\underset{\sim}{z}} = A(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}_1) = A\bar{\underset{\sim}{x}} - \underset{\sim}{b}$

is an equivalent version of the test.

Note that $T_z^2 = T^2$ $\left(A\bar{\underset{\sim}{z}} = A(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}_1) = A\bar{\underset{\sim}{x}} - \underset{\sim}{b} \right)$

Repeated measurement designs

$$H_0: \mu_1 = \mu_2 = \dots = \mu_d \;(=\mu) \iff \underset{\sim}{\mu} = \mu \underset{\sim}{1}_d \iff \underset{\sim}{\mu} \in \mathcal{R}(\underset{\sim}{1}_d)$$

$\Rightarrow \exists \underset{(d-1)\times d}{A} : \underset{\sim}{1}_d = \mathcal{N}(A)$ cf. B 2.3 $\Rightarrow \mu A \underset{\sim}{1}_d = \underset{\sim}{0} \Rightarrow A\underset{\sim}{\mu} = \underset{\sim}{0}$

$\mathcal{N}(A) = \mathcal{R}(\underset{\sim}{1}_d)$ iff B 2.3 $\iff \mathcal{R}(A^T)^{\perp} = \mathcal{R}(\underset{\sim}{1}_d)$ cf. B 2.1

A is <u>not</u> unique ; choosing

- $H_0: \mu_1 - \mu_2 = \mu_2 - \mu_3 = \dots = \mu_{d-1} - \mu_d = 0$, we get

$$\begin{bmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{bmatrix} \underset{\sim}{\mu} = \underset{\sim}{0} \iff C_1 \underset{\sim}{\mu} = \underset{\sim}{0} , \quad \text{rank } C_1 = d-1$$

- $H_0: \mu_1 - \mu_d = \mu_2 - \mu_d = \dots = \mu_{d-1} - \mu_d = 0$, we get

$$\begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & 1 & -1 \end{bmatrix} \underset{\sim}{\mu} = \underset{\sim}{0} \iff C_2 \underset{\sim}{\mu} = \underset{\sim}{0} , \quad \text{rank } C_2 = d-1$$

Note that

$$C_1 = \begin{bmatrix} 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & & 0 & 1 \end{bmatrix} C_2 = F_1 C_2 \quad \text{and} \quad C_2 = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} C_1 = F_2 C_1$$

F_1 and F_2 are invertible $(d-1)\times(d-1)$ matrices , $F_1 F_2 = I_{d-1}$

Let $A\underset{\sim}{\mu} = \underset{\sim}{0}$ be any representation of H_0.
We can then find an invertible $(d-1)\times(d-1)$ matrix F, so $A = FC_1$, as the rows of A and the rows of C_1 both span $\mathcal{R}(\underset{\sim}{1}_d)^{\perp}$
$= \mathcal{N}(\underset{\sim}{1}_d^T) = \{ \underset{\sim}{x} \mid \underset{\sim}{1}_d^T \underset{\sim}{x} = 0 \} = \{ \underset{\sim}{x} \mid \sum_{j=1}^{d} x_j = 0 \}$
cf. B 2.1

Note that the Hotelling's T^2 test for $H_0: A\underset{\sim}{\mu} = \underset{\sim}{0}$ using the observations $\underset{\sim}{x}_i$ is equivalent to Hotelling's T^2 test for $H_0: F\underset{\sim}{\nu} = \underset{\sim}{0}$, $\underset{\sim}{\nu} = C_1 \underset{\sim}{\mu}$ using $\underset{\sim}{y}_i = C_1 \underset{\sim}{x}_i$ as the observed vector.

Ex.

EXAMPLE 3.2 Rao [1948] introduced the now famous example of testing whether the thickness of cork borings on trees was the same in the north, east, south and west directions; thus $d = 4$ and H_0 is $\mu_1 = \mu_2 = \mu_3 = \mu_4$. Observations $\mathbf{x}_i$ on 28 trees are given in Table 3.3, for which

$$\bar{\mathbf{x}} = \begin{array}{c} \begin{pmatrix} 50.5357 \\ 46.1786 \\ 49.6786 \\ 45.1786 \end{pmatrix} \end{array} \begin{array}{l} \mathrm{N} \\ \mathrm{E} \\ \mathrm{S} \\ \mathrm{W} \end{array} \quad \text{and} \quad \mathbf{Q}_x = \begin{pmatrix} 7840.96 & 6041.32 & 7787.82 & 6109.32 \\ - & 5938.11 & 6184.61 & 4627.11 \\ - & - & 9450.11 & 7007.61 \\ - & - & - & 6102.11 \end{pmatrix}.$$

TABLE 3.3 Weight of Cork Borings (in Centigrams) in Four Directions for 28 Trees[a]

N	E	S	W	N	E	S	W
72	66	76	77	91	79	100	75
60	53	66	63	56	68	47	50
56	57	64	58	79	65	70	61
41	29	36	38	81	80	68	58
32	32	35	36	78	55	67	60
30	35	34	26	46	38	37	38
39	39	31	27	39	35	34	37
42	43	31	25	32	30	30	32
37	40	31	25	60	50	67	54
33	29	27	36	35	37	48	39
32	30	34	28	39	36	39	31
63	45	74	63	50	34	37	40
54	46	60	52	43	37	39	50
47	51	52	43	48	54	57	43

[a]From Rao [1948: Table 1], by permission of the Biometrika Trustees.

EXAMPLE 3.3 In Example 3.2, Rao was particularly interested in cork differences relating to the contrast north–south versus east–west ($\mu_1 + \mu_3 - \mu_2 - \mu_4$), to which he added the contrasts south versus west ($\mu_3 - \mu_4$) and north versus south ($\mu_1 - \mu_3$). This choice of contrasts leads to the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Since testing $H_0: \mathbf{A}\boldsymbol{\mu} = \mathbf{0}$ is equivalent to testing $\mu_1 = \mu_2 = \mu_3 = \mu_4$, the value of the test statistic F_0 is the same as that in Example 3.2, namely, 6.402. By the same token, the measures of skewness and kurtosis, $b_{1,3}$ and $b_{2,3}$, are also the same, as $\mathbf{A}$ is related to $\mathbf{C}_1$ by a nonsingular transformation.

$$A = F C_1 \quad \Leftrightarrow \quad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Test for symmetry

$\underset{\sim}{x}_j \sim N_{2d}$, $j = 1, \ldots, n$, independent , $2d < n-1$

$H_0 : \mu_i = \mu_{i+d}$, $i = 1, \ldots, d$

$\Leftrightarrow [I_d \;\; -I_d]\underset{\sim}{\mu} = \underset{\sim}{0} \;\Leftrightarrow\; A\underset{\sim}{\mu} = \underset{\sim}{0}$, rank $A = d$

Let $\bar{\underset{\sim}{x}} = \begin{array}{cc} \begin{bmatrix} \bar{\underset{\sim}{x}}_1 \\ \bar{\underset{\sim}{x}}_2 \end{bmatrix} & \begin{array}{c} d \\ d \end{array} \end{array}$, $S = \begin{array}{cc} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} & \begin{array}{c} d \\ d \end{array} \\ \begin{array}{cc} d & d \end{array} & \end{array}$

$$T^2 = n(A\bar{\underset{\sim}{x}})^T (ASA^T)^{-1} A\bar{\underset{\sim}{x}} \sim T^2(d, n-1)$$

$$= n(\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)^T (S_{11} - S_{21} - S_{12} + S_{22})^{-1} (\bar{\underset{\sim}{x}}_1 - \bar{\underset{\sim}{x}}_2)$$

Test of $H_0 : \underset{\sim}{\mu} = K\underset{\sim}{\beta}$

K $d \times k$, rank $K = k < d$

$P = K(K^TK)^{-1}K^T$ is representing the orthogonal projektion of R^d on $\mathcal{R}(K)$, rank $P = k$

$I_d - P$ is representing the orthogonal projektion on $\mathcal{R}(K)^\perp$, i.e. $\underset{\sim}{\mu}$ must fulfill $(I_d - P)\underset{\sim}{\mu} = \underset{\sim}{0}$

The least complex version of $H_0 : A\underset{\sim}{\mu} = \underset{\sim}{0}$ can therefore be found by choosing the rows of A as $d-k$ linear independent rows from $I_d - P$.

With independent observations $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \ldots, n$, we can use the test statistic

$T^2 = n\bar{\underset{\sim}{x}}^T A^T (ASA^T)^{-1} A\bar{\underset{\sim}{x}} \sim T^2(d-k, n-1)$ when H_0 is true

Minimization technique

Theorem 3.3

$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$ indep., $d \le n-1$

H_0: $\underset{\sim}{\mu} \in V$, V p-dim. subspace of R^d

When H_0 is true

$$\min_{\underset{\sim}{\mu} \in V} n(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu})^T S^{-1} (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}) \sim T^2(d-p, n-1)$$

Proof

Let $p_0 = d - p$ and

let P represent the orthogonal projection of R^d on V.

Let A consist of p_0 linear independent rows from $I_d - P$, hence A $p_0 \times d$, rank $A = p_0$

$V = \mathcal{R}(P) = \mathcal{R}(I_d - P)^\perp = \mathcal{R}(A^T)^\perp = \mathcal{N}(A)$ cf. B2.1

$\underset{\sim}{\mu} \in V \Leftrightarrow A\underset{\sim}{\mu} = \underset{\sim}{0}$

$S \underset{a.s.}{>} 0 \Rightarrow S^{-1} \underset{a.s.}{>} 0 \Rightarrow \exists R : S^{-1} \underset{a.s.}{=} R^T R$, where R is non-singular, cf. A5.3

Let $\underset{\sim}{z} = R\bar{\underset{\sim}{x}}$, $\underset{\sim}{\theta} = R\underset{\sim}{\mu}$

$$\begin{aligned}(\bar{\underset{\sim}{x}} - \underset{\sim}{\mu})^T S^{-1} (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}) &= (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu})^T R^T R (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}) \\ &= (R\bar{\underset{\sim}{x}} - R\underset{\sim}{\mu})^T (R\bar{\underset{\sim}{x}} - R\underset{\sim}{\mu}) \\ &= (\underset{\sim}{z} - \underset{\sim}{\theta})^T (\underset{\sim}{z} - \underset{\sim}{\theta}) = \|\underset{\sim}{z} - \underset{\sim}{\theta}\|^2\end{aligned}$$

$$\begin{aligned}A\underset{\sim}{\mu} = \underset{\sim}{0} &\Leftrightarrow AR^{-1}R\underset{\sim}{\mu} = \underset{\sim}{0} \Leftrightarrow AR^{-1}\underset{\sim}{\theta} = \underset{\sim}{0} \\ &\Leftrightarrow A_1\underset{\sim}{\theta} = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{\theta} \in \mathcal{N}(A_1) \quad (A_1 = AR^{-1})\end{aligned}$$

The minimization problem is now turned into determining $\min_{\underset{\sim}{\theta} \in \mathcal{N}(A_1)} \| \underset{\sim}{z} - \underset{\sim}{\theta} \|^2$, which according to the least squares principle has the solution

$$\underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}} = P_1 \underset{\sim}{z}, \quad P_1 = I_d - A_1^T (A_1 A_1^T)^{-1} A_1 \quad \text{cf. B 2.2}$$
$$= I_d - P_2$$

$$\underset{\sim}{z} - \hat{\underset{\sim}{\theta}} = \underset{\sim}{z} - P_1 \underset{\sim}{z} = (I_d - P_1) \underset{\sim}{z} = P_2 \underset{\sim}{z}$$

$$n \| \underset{\sim}{z} - \hat{\underset{\sim}{\theta}} \|^2 = n (\underset{\sim}{z} - \hat{\underset{\sim}{\theta}})^T (\underset{\sim}{z} - \hat{\underset{\sim}{\theta}}) = n \underset{\sim}{z}^T P_2^T P_2 \underset{\sim}{z} = n \underset{\sim}{z}^T P_2 \underset{\sim}{z}$$
$$= n \underset{\sim}{z}^T A_1^T (A_1 A_1^T)^{-1} A_1 \underset{\sim}{z} = n (A_1 \underset{\sim}{z})^T (A_1 A_1^T)^{-1} A_1 \underset{\sim}{z}$$
$$= n (A R^{-1} R \bar{\underset{\sim}{x}})^T (A R^{-1} R^{-1T} A^T)^{-1} A R^{-1} R \bar{\underset{\sim}{x}}$$
$$= n (A \bar{\underset{\sim}{x}})^T (A S A^T)^{-1} A \bar{\underset{\sim}{x}}$$
$$= T^2 \sim T^2(p_0, n-1) = T^2(d-p, n-1)$$

Corollary

$$\min_{A \underset{\sim}{\mu} = \underset{\sim}{b}} n (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu})^T S^{-1} (\bar{\underset{\sim}{x}} - \underset{\sim}{\mu}) \sim T^2(\text{rank } A, n-1)$$

Proof

$A \underset{\sim}{\mu} = \underset{\sim}{b} \iff A \underset{\sim}{\chi} = \underset{\sim}{0}$ where $\underset{\sim}{\chi} = \underset{\sim}{\mu} - \underset{\sim}{\mu}_1$ cf. section 3.4.1

$$\underset{\sim}{u}_i = \underset{\sim}{x}_i - \underset{\sim}{\mu}_1 \Rightarrow \bar{\underset{\sim}{x}} - \underset{\sim}{\mu} = (\bar{\underset{\sim}{u}} + \underset{\sim}{\mu}_1) - \underset{\sim}{\mu} = \bar{\underset{\sim}{u}} - (\underset{\sim}{\mu} - \underset{\sim}{\mu}_1)$$
$$= \bar{\underset{\sim}{u}} - \underset{\sim}{\chi}$$

S unchanged, cf. formula (1.11)

Theorem 3.3 now leads to

$$\min_{A \underset{\sim}{\chi} = \underset{\sim}{0}} n (\bar{\underset{\sim}{u}} - \underset{\sim}{\chi})^T S^{-1} (\bar{\underset{\sim}{u}} - \underset{\sim}{\chi}) \sim T^2(\text{rank } A, n-1)$$