Problems 11

Comments and hints are additional to those given in the book p. 618.

3.12 Choose an orthogonal transformation $y_i = T^T \underset{\sim}{x_i}$ so that Σ becomes diagonalized, i.e.

$y_i \sim N_d(T^T \underset{\sim}{\mu}, \Lambda)$, $\Lambda = T^T \Sigma T$

A suitable choice of T leads to

$$\Lambda = \sigma^2 \begin{bmatrix} 1+(d-1)\rho & 0 & \cdots & 0 \\ 0 & 1-\rho & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-\rho \end{bmatrix} = \begin{bmatrix} a+(d-1)b & 0 & \cdots & 0 \\ 0 & a-b & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a-b \end{bmatrix}$$

Determine the likelihood function corresponding to the transformed observations:

$$\begin{aligned} \ln L_Y(T^T\hat{\mu}, a, b) = &- \frac{nd}{2}\ln(2\pi) \\ &- \frac{n}{2}\left(\ln(a+(d-1)b) + (d-1)\ln(a-b)\right) \\ &- \frac{1}{2}\left(\frac{q_1}{a+(d-1)b} + \frac{q_2}{a-b}\right), \end{aligned}$$

where $q_1 = q_{11}^Y$ and $q_2 = \sum_{j=2}^{d} q_{jj}^Y$

Differentiate w r t a and b, and solve the equation system $\frac{\partial \ln L}{\partial a} = 0$, $\frac{\partial \ln L}{\partial b} = 0$.

Calculation technique: At first find q_1 and q_2 expressed by a and b. Then solve the new equation system w r t a and b:

$$a = \frac{1}{nd}(q_1 + q_2), \quad b = \frac{1}{nd}\left(q_1 - \frac{1}{d-1}q_2\right)$$

cont.

3.12 continued

Note that $q_1 + q_2 = \mathrm{tr}\, Q_Y = \mathrm{tr}\, Q$, as trace is invariant towards similarity transformations (follows from A 1.1 (b)), and that

$$q_1 - \frac{1}{d-1} q_2 = \frac{d}{d-1} q_1 - \frac{1}{d-1} \mathrm{tr}\, Q_Y = \frac{1}{d-1}\left(\sum_j \sum_k \underset{\sim}{1}_d^T Q \underset{\sim}{1}_d - \mathrm{tr}\, Q \right)$$

Determine a and b expressed by the elements of S $\quad (Q = (n-1) S)$:

$$\hat{a} = \frac{n-1}{n} \frac{1}{d} \sum_j s_{jj} \quad , \quad \hat{b} = \frac{n-1}{n} \frac{1}{d(d-1)} \sum_j \sum_{k \neq j} s_{jk}$$

Alternatively

Determine the likelihood function corresponding to the $\underset{\sim}{x}$ observations :

$$\ln L(\underset{\sim}{\mu}, a, b) = -\frac{nd}{2} \ln(2\pi)$$
$$- \frac{n}{2}\left((d-1) \ln(a-b) + \ln(a + (d-1) b) \right)$$
$$- \frac{1}{2} \frac{1}{(a-b)(a+(d-1)b)} \left(a + (d-2) q_0 - b q \right),$$

where $q_0 = \sum_j q_{jj}$ and $q = \sum_j \sum_{k \neq j} q_{jk}$

Use computer algebra for differentiation of $\ln L$ w r t a and b and for solving the equation system $\frac{\partial \ln L}{\partial a} = 0$, $\frac{\partial \ln L}{\partial b} = 0$:

$$\hat{a} = \frac{1}{nd} q_0 \quad , \quad \hat{b} = \frac{1}{nd(d-1)} q$$

3.13 Calculate $\mathrm{tr}\left((\Lambda(\hat{a}, \hat{b}))^{-1} Q_Y \right)$ $\quad (= nd)$

Determine $L(\hat{\underset{\sim}{\mu}}, \hat{a}, \hat{b})$ $\quad (= L_Y(T^T \hat{\underset{\sim}{\mu}}, \hat{a}, \hat{b}))$

Show that $$\ell_6^{\frac{2}{n}} = \frac{\left(\frac{n-1}{n}\right)^d \det S}{(\hat{\sigma}^2)^d (1-\hat{\rho})^{d-1} (1 + (d-1)\hat{\rho})}$$

Show also that $$\ell_6^{\frac{2}{n}} = \frac{(d-1)^{d-1} d^d \det S}{\left(d \,\mathrm{tr}\, S - \underset{\sim}{1}_d^T S \underset{\sim}{1}_d\right)^{d-1} \underset{\sim}{1}_d^T S \underset{\sim}{1}_d}$$

Alternatively (cf. ex. 3.12)

Calculate $\mathrm{tr}\left((\Sigma(\hat{a}, \hat{b}))^{-1} Q \right)$ $\quad (= nd)$

Other calculations no change