MAT4 / MA

Problems 4

Comments and hints are additional to those given in the book p. 616.

2.9 Let X be a stochastic matrix $(n \times d)$ and T a matrix $(n \times d)$ of real variables.

Definition of moment generating function:

$M_X(T) = E[\operatorname{etr}(T^T X)]$

Is this definition compatible with the definition of $M_{\underset{\sim}{x}}(\underset{\sim}{t})$ given earlier?

Show also that the usual features of moment generating functions still are valid, i.e.

(1) $M_{AX+B}(T) = \operatorname{etr}(T^T B)\, M_X(A^T T)$

(2) X, Y indep. $\Rightarrow$ $M_{X+Y}(T) = M_X(T)\, M_Y(T)$

Alternative expressions for $M_X(T)$:

$$M_X(T) = E\left[\exp \sum_k (T^T X)_{kk}\right] = E\left[\exp \sum_k \sum_j t^T_{kj} x_{jk}\right]$$

$$= E\left[\exp \sum_j \sum_k t_{jk} x_{jk}\right]$$

For a symmetric stochastic matrix, say W, we change to

$$M_W(U) = E\left[\exp \sum_j \sum_k u_{jk} w_{jk}\right], \text{ where } \begin{array}{l} u_{jj} = t_{jj} \\ u_{jk} = \frac{1}{2}(t_{jk} + t_{kj}),\ k \neq j \end{array}$$

$$= E[\operatorname{etr}(UW)]$$

Note that U is chosen symmetric. The form $M_W(U)$ should be used in the present exercise.

cont.

2.9 continued

In the text line 5 there is missing $\Leftrightarrow M_W(U)$ in front of the equation sign.

Another option is

$$M_W(S) = E\left[\exp \sum_j \sum_{k \geq j} s_{jk} w_{jk}\right], \quad \text{where } s_{jj} = t_{jj}$$
$$s_{jk} = t_{jk} + t_{kj}, \quad k > j$$

Note that S is upper triangular. This form with $S := T$ is mentioned in the text but should not be used here.

2.10 Calculate EW in two different ways:

(1) As $E[X^T X]$ and (2) as $E[\sum_i \underset{\sim}{x}_i \underset{\sim}{x}_i^T]$

2.11 The density function for a $\chi^2(m)$ distribution is

$$f(x) = \frac{1}{\Gamma(\frac{m}{2})} 2^{-\frac{m}{2}} x^{\frac{m-2}{2}} e^{-\frac{x}{2}}, \quad 0 < x < \infty$$

Derive the corresponding density for a $\sigma^2\chi^2(m)$ distribution.

2.12 I the outline solution line 2 and 3 read $(1-2\sigma^2 t)^{-\frac{m}{2}}$ instead of $(1-2t)^{-\frac{m}{2}}$. This result can be found in different ways:

1st method

$$M_W(U) = (\det(I_d - 2U\Sigma))^{-\frac{m}{2}} \Rightarrow M_W(u) = (\det(1-2u\sigma^2))^{-\frac{m}{2}}$$
$$\Leftrightarrow M_W(t) = (\det(1-2\sigma^2 t))^{-\frac{m}{2}}$$

2nd method

$$f(w) = \frac{1}{\Gamma(\frac{m}{2})}(2\sigma^2)^{-\frac{m}{2}} w^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}}, \quad 0 < w < \infty$$

$$M_W(t) = \int_0^\infty e^{tw} f(w)\, dw$$

$$= \int_0^\infty \frac{1}{\Gamma(\frac{m}{2})}(2\sigma^2)^{-\frac{m}{2}} w^{\frac{m-2}{2}} e^{-\frac{(1-2\sigma^2 t)w}{2\sigma^2}}\, dw, \quad u = (1-2\sigma^2 t)w, \; du = (1-2\sigma^2 t)\,dw$$

cont.

2.12 continued

$$M_w(t) = (1-2\sigma^2 t)^{-1-\frac{m-2}{2}} \int_0^\infty \frac{1}{\Gamma(\frac{m}{2})} (2\sigma^2)^{-\frac{m}{2}} u^{\frac{m-2}{2}} e^{-\frac{u}{2\sigma^2}} du$$

$$= (1-2\sigma^2 t)^{-\frac{m}{2}}$$

3rd method

$w = \sum x_i^2$, $x_i \sim N(0, \sigma^2)$, $i = 1, \dots, m$, independant

$$M_{x_i^2}(t) = \int_{-\infty}^\infty e^{tx_i^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_i^2}{2\sigma^2}} dx_i = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(1-2\sigma^2 t)x_i^2}{2\sigma^2}} dx_i$$

$$= (1-2\sigma^2 t)^{-\frac{1}{2}} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\,\sigma (1-2\sigma^2 t)^{-\frac{1}{2}}} e^{-\frac{x_i^2}{2\sigma^2(1-2\sigma^2 t)^{-1}}} dx_i$$

$$= (1-2\sigma^2 t)^{-\frac{1}{2}}$$

$$M_w(t) = \prod_{i=1}^m (1-2\sigma^2 t)^{-\frac{1}{2}} = (1-2\sigma^2 t)^{-\frac{m}{2}}$$

a It is convenient to use the following rule:

$$\det(I_n + AB) = \det(I_k + BA)$$

The rule can be established by use of app. A 3.2. Let A and B be $n \times k$ and $k \times n$:

$$\begin{vmatrix} I_n & A \\ -B & I_k \end{vmatrix} = \begin{cases} \det I_k \det(I_n - A I_k^{-1}(-B)) \\ \det I_n \det(I_k - (-B) I_n^{-1} A) \end{cases} = \begin{cases} \det(I_n + AB) \\ \det(I_k + BA) \end{cases}$$

b Look at $M_{W_{11}}(U_{11}) = M_{\begin{bmatrix} W_{11} & 0 \\ 0 & 0 \end{bmatrix}}\left(\begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix}\right)$, and show that $W_{11} \sim W_r(m, \Sigma_{11})$

Now look at $M_{W_{11}+W_{22}}(U_{11}, U_{22}) = M_{\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix}}\left(\begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix}\right)$,

and show that $\Sigma_{12} = 0 \Leftrightarrow W_{11}$ and W_{22} independent

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