Problems 8

Comments and hints are additional to those given in the book p. 617.

3.1 Do not use the outline solution in the book

3.2 In the outline solution in the book read S instead of S^{-1}.

3.3 Correction to the text last line : 2.1 (iv) → 2.1 (v)

Corrections to the outline solution in the book

- line 5 : $(\bar{\underset{\sim}{x}}_{\cdot 1}, \bar{\underset{\sim}{x}}_{\cdot 2}, \dots, \bar{\underset{\sim}{x}}_{\cdot d}) \to (\bar{x}_{\cdot 1}, \bar{x}_{\cdot 2}, \dots, \bar{x}_{\cdot d})$
- line 6 : $= A \to = \sigma^2 A$
- line 6 : $\chi^2_{d-1} \to \sigma^2 \chi^2_{d-1}$
- line 8 : $Q_H \to Q_E$
- line 9 : $\chi^2_{(n-1)(d-1)} \to \sigma^2 \chi^2_{(n-1)(d-1)}$
- line 9 : $= A_0 \to = \sigma^2 A_0$

Comments to the outline solution in the book :

(a) Calculate the covariance (result 0) and show that Q_H and Q_E are independent

(b) Let $\underset{\sim}{\bar{z}} = \sqrt{\frac{n}{1-\rho}}\,(\underset{\sim}{\bar{x}} - \mu \underset{\sim}{1}_d)$

$$\sim N_d\Big(\underbrace{\sqrt{\tfrac{n}{1-\rho}}\,(\underset{\sim}{\mu} - \mu \underset{\sim}{1}_d)}_{=\underset{\sim}{0},\ H_0 \text{ true}},\ \underbrace{\frac{n}{1-\rho}\,\frac{\Sigma}{n}}_{=\frac{\Sigma}{1-\rho}}\Big)$$

Determine A, so $\frac{1}{1-\rho} Q_H = \underset{\sim}{\bar{z}}^T A \underset{\sim}{\bar{z}}$

(Result : $A = I_d - \frac{1}{d} \underset{\sim}{1}_d \underset{\sim}{1}_d^T$)

Using A 6.5 show that

$$\frac{1}{1-\rho} Q_H \sim \sigma^2 \chi^2(d-1) \text{ when } H_0 \text{ is true}$$

(c) Let $\mathbf{z} = \sqrt{\frac{1}{1-\rho}}\,(\mathbf{x} - \mu \mathbf{1}_{nd}) \sim N_{nd}\Big(\underbrace{\sqrt{\tfrac{1}{1-\rho}}(\boldsymbol{\mu} - \mu \mathbf{1}_{nd})}_{=\,\mathbf{0},\ H_0 \text{ true}},\ \underbrace{\tfrac{1}{1-\rho} I_n \otimes \Sigma}_{=\,\Sigma_0}\Big)$

$(\boldsymbol{\mu} := [\boldsymbol{\mu}^T\ \boldsymbol{\mu}^T \dots \boldsymbol{\mu}^T]^T)$

Show that

$$\frac{1}{1-\rho} Q_E \sim \sigma^2 \chi^2((n-1)(d-1)) \quad \text{when } H_0 \text{ is true}$$

Intermediate result:

$$\frac{1}{1-\rho} Q_E = \mathbf{z}^T A_0 \mathbf{z}, \quad \text{where}$$

$$A_0 = I_{nd} - \frac{1}{d} I_n \otimes \mathbf{1}_d \mathbf{1}_d^T - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \otimes I_d + \frac{1}{nd} \mathbf{1}_{nd} \mathbf{1}_{nd}^T$$

(d) Show that

$$\frac{(n-1)\,Q_H}{Q_E} \sim F(d-1, (n-1)(d-1)) \quad \text{when } H_0 \text{ is true}$$

(e) Show that

$$E Q_H = (d-1)\sigma^2(1-\rho) + n \sum_{j=1}^{d} (\mu_j - \bar{\mu})^2,$$

$$\text{where } \bar{\mu} = \frac{1}{d} \sum_{j=1}^{d} \mu_j$$

Remember the formula in ex. 1.4:

$$E[\mathbf{x}^T A \mathbf{x}] = \operatorname{tr}(A\Sigma) + \boldsymbol{\theta}^T A \boldsymbol{\theta}, \quad E\mathbf{x} = \boldsymbol{\theta}, \quad \operatorname{Var} \mathbf{x} = \Sigma$$

Intermediate results:

$A\Sigma = \sigma^2(1-\rho)A$ (by indirect use of the result in ex. 1.9 modified)

$$(\boldsymbol{\mu} - \mu \mathbf{1}_d)^T A (\boldsymbol{\mu} - \mu \mathbf{1}_d) = \sum_{j=1}^{d} (\mu_j - \bar{\mu})^2$$

Note that

$$E[(n-1)Q_H] = (n-1)(d-1)\sigma^2(1-\rho) \quad \text{when } H_0 \text{ is true}$$

(f) Show that

$$E Q_E = (n-1)(d-1)\sigma^2(1-\rho)$$

Intermediate results :

$$A_0 \Sigma_0 = \sigma^2 A_0$$

$$A_0 (\underset{\sim}{\mu} - \mu \underset{\sim}{1}_{nd}) = \underset{\sim}{0}$$

Note that $E\,Q_E$ has the same value, H_0 true or false

(g) Determine the critical area for the test statistic used for testing H_0