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1 $$\mathrm{Cov}[\underline{X},\underline{Y}] = E[(\underline{X}-E\underline{X})(\underline{Y}-E\underline{Y})^T]$$

$$= E[\underline{X}\,\underline{Y}^T - \underline{X}\,E\underline{Y}^T - (E\underline{X})\,\underline{Y}^T + E\underline{X}\,E\underline{Y}^T]$$

$$= E[\underline{X}\,\underline{Y}^T] - E\underline{X}\,E\underline{Y}^T - E\underline{X}\,E\underline{Y}^T + E\underline{X}\,E\underline{Y}^T$$

$$= E[\underline{X}\,\underline{Y}^T] - E\underline{X}\,E\underline{Y}^T$$

2 $$\mathrm{Cov}[A\underline{X}, B\underline{Y}] = E[(A\underline{X} - E[A\underline{X}])(B\underline{Y} - E[B\underline{Y}])^T]$$

$$= E[A(\underline{X}-E\underline{X})(B(\underline{Y}-E\underline{Y}))^T]$$

$$= A\,E[(\underline{X}-E\underline{X})(\underline{Y}-E\underline{Y})^T]\,B^T$$

$$= A\,\mathrm{Cov}[\underline{X},\underline{Y}]\,B^T$$

3 $$\mathrm{Var}[\underline{Y}+\underline{b}] = \mathrm{Cov}[\underline{Y}+\underline{b}, \underline{Y}+\underline{b}]$$

$$= E[(\underline{Y}+\underline{b}-E[\underline{Y}+\underline{b}])(\underline{Y}+\underline{b}-E[\underline{Y}+\underline{b}])^T]$$

$$= E[(\underline{Y}-E\underline{Y})(\underline{Y}-E\underline{Y})^T]$$

$$= \mathrm{Cov}[\underline{Y},\underline{Y}]$$

$$= \mathrm{Var}\,\underline{Y}$$

4 $$M_{A\underline{Y}+\underline{b}}(\underline{t}) = E[\exp(\underline{t}^T(A\underline{Y}+\underline{b}))]$$

$$= E[\exp(\underline{t}^T A\underline{Y} + \underline{t}^T\underline{b})]$$

$$= \exp(\underline{t}^T\underline{b})\,E[\exp((A^T\underline{t})^T\underline{Y})]$$

$$= \exp(\underline{t}^T\underline{b})\,M_{\underline{Y}}(A^T\underline{t})$$

5 $\underline{X}, \underline{Y}$ unabh.

$$M_{\underline{X}+\underline{Y}}(\underline{t}) = E[\exp(\underline{t}^T(\underline{X}+\underline{Y}))]$$

$$= E[\exp(\underline{t}^T\underline{X} + \underline{t}^T\underline{Y})]$$

$$= E[\exp(\underline{t}^T\underline{X})]\,E[\exp(\underline{t}^T\underline{Y})]$$

$$= M_{\underline{X}}(\underline{t})\,M_{\underline{Y}}(\underline{t})$$

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6 C ortogonal, dvs. $C^T C = I$

i $C^T C = I \Rightarrow C C^T = I$, jf. komp. i lin. alg. s. 15, dvs. systemet af rækkevektorer i C er også ortonormeret

ii $C^T C = C C^T = I \Rightarrow C^{-1} = C^T$, jf def. af C^{-1}

iii $C^T C = I \Rightarrow \det(C^T C) = \det I$

$\Rightarrow \det C^T \det C = 1$

$\Rightarrow (\det C)^2 = 1$

$\Rightarrow \det C = \pm 1$

iv A og B ortogonale

$(AB)^T (AB) = B^T A^T A B = B^T I B = B^T B = I$

$\Rightarrow AB$ ortogonal

7 A $n \times n$ symmetrisk *

C ortogonal, $C^T A C = \Lambda = \ulcorner \lambda_1 \ldots \lambda_n \lrcorner$ $(C^T = C^{-1})$

Betragt $\underline{x}^T A \underline{x}$ for alle $\underline{x} \in R^n$

Sæt $y = C^{-1}\underline{x} \Leftrightarrow \underline{x} = C y$, bemærk $\underline{x} \neq \underline{0} \Leftrightarrow y \neq \underline{0}$

$\underline{x}^T A \underline{x} = (Cy)^T A (Cy = y^T C^T A C y = y^T \Lambda y = \sum_i \lambda_i y_i^2$

i $\underline{x} = C \underline{e}_j \Rightarrow y = e_j \Rightarrow \sum_i \lambda_i y_i^2 = \lambda_j$

dvs. $\underline{x}^T A \underline{x} > 0$ for alle $\underline{x} \Rightarrow \lambda_j > 0$, $j = 1, \ldots, n$

ii $\forall \underline{x} \neq \underline{0}$: $\lambda_i > 0 \Rightarrow \lambda_i y_i^2 > 0 \Rightarrow \sum_i \lambda_i y_i^2 > 0$

$\Leftrightarrow \underline{x}^T A \underline{x} > 0$

altså $\forall \underline{x} \neq \underline{0} : \underline{x}^T A \underline{x} > 0 \Leftrightarrow \lambda_i > 0$, $i = 1, \ldots, n$

* Spektralsætningen siger bl. a., at enhver symmetrisk matrix kan diagonaliseres ved en ortogonal matrix.

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8 $\quad A$ positiv definit, C ortogonal, $C^{-1}AC = \Lambda$

$\Lambda = \ulcorner \lambda_1 \ \dots \ \lambda_n \lrcorner = \left(\ulcorner \lambda_1^{\frac{1}{2}} \ \dots \ \lambda_n^{\frac{1}{2}} \lrcorner \right)^2$,

idet $\lambda_i > 0$ for alle i, jf. opg. 7

Sæt $\Lambda^{\frac{1}{2}} = \ulcorner \lambda_1^{\frac{1}{2}} \ \dots \ \lambda_n^{\frac{1}{2}} \lrcorner$, dvs. $(\Lambda^{\frac{1}{2}})^2 = \Lambda$

Sæt $A^{\frac{1}{2}} = C\Lambda^{\frac{1}{2}}C^{-1}$, hvorefter

$(A^{\frac{1}{2}})^2 = C\Lambda^{\frac{1}{2}}C^{-1}C\Lambda^{\frac{1}{2}}C^{-1} = C\Lambda C^{-1} = A$

9. $\det \Sigma = \det(\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}) = (\det \Sigma^{\frac{1}{2}})^2 \Rightarrow \det \Sigma^{\frac{1}{2}} = (\det \Sigma)^{\frac{1}{2}}$

10 $\quad \int_{R^n} f_{\underline{Y}}(y_1, \dots, y_n)\, dV$ noteres $\int_{R^n} f_{\underline{Y}}(\underline{y})\, d\Omega$

Indfør variabeltransformation $\underline{y} = \Sigma^{\frac{1}{2}}\underline{x} + \underline{\mu}$ med Jacobideterminant $J(\underline{x}) = \det \Sigma^{\frac{1}{2}}$:

$$\int_{R^n} f_{\underline{Y}}(\underline{y})\, dV = \int_{R^n} f_{\underline{Y}}(\Sigma^{\frac{1}{2}}\underline{x} + \underline{\mu})\, |\det \Sigma^{\frac{1}{2}}|\, d\Omega$$

$$= \int_{R^n} f_{\underline{X}}(\underline{x})\, d\Omega \quad \text{, jf. note s. 3}$$

$$= \prod_{i=1}^{n} \int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} dx_i = \prod_{i=1}^{n} 1 = 1$$

11 $\quad$ i $\quad \underline{Y} \sim N_n(\underline{\mu}, \Sigma) \Rightarrow M_{\underline{Y}}(\underline{t}) = \exp(\underline{t}^T\underline{\mu} + \frac{1}{2}\underline{t}^T\Sigma\underline{t})$

$\Rightarrow \forall \underline{\ell} \neq \underline{0} : M_{\underline{\ell}^T\underline{Y}}(t) = M_{\underline{Y}}(\underline{\ell}t)$, jf. opg. 4

$= \exp(\underline{\ell}^T\underline{\mu}t, \frac{1}{2}\underline{\ell}^T\Sigma\underline{\ell}\, t^2)$

$\Leftrightarrow \forall \underline{\ell} \neq \underline{0} : \underline{\ell}^T\underline{\mu} \sim N(\underline{\ell}^T\underline{\mu}, \frac{1}{2}\underline{\ell}^T\Sigma\underline{\ell})$

ii $\quad \forall \underline{\ell} \neq \underline{0} : \underline{\ell}^T\underline{Y} \sim N(\underline{\ell}^T\underline{\mu}, \frac{1}{2}\underline{\ell}^T\Sigma\underline{\ell})$

$\Rightarrow M_{\underline{\ell}^T\underline{Y}}(t) = \exp(\underline{\ell}^T\underline{\mu}t + \frac{1}{2}\underline{\ell}^T\Sigma\underline{\ell}\, t^2)$, sæt $\underline{\ell} = \underline{t}$ og $t = 1$

$\Rightarrow M_{\underline{t}^T\underline{Y}}(1) = \exp(\underline{t}^T\underline{\mu}\, 1 + \frac{1}{2}\underline{t}^T\Sigma\underline{t}\, 1^2)$

$\Leftrightarrow M_{\underline{Y}}(\underline{t}) = \exp(\underline{t}^T\underline{\mu} + \frac{1}{2}\underline{t}^T\Sigma\underline{t})$, jf. opg. 4

$\Leftrightarrow \underline{Y} \sim N_n(\underline{\mu}, \Sigma)$

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12 A $n\times n$ pos. def. , C $p\times n$, rang $C = p$

$(CAC^T)^T = (C^T)^T A^T C^T = CAC^T$, dvs. CAC^T sym.

$\forall \underline{x} \neq \underline{0}$: $\underline{x}^T (CAC^T) \underline{x} = (C^T\underline{x})^T A C^T \underline{x} = \underline{y}^T A \underline{y} > 0$,

idet $\underline{y} \neq \underline{0} \Leftrightarrow C^T\underline{x} \neq \underline{0} \Leftrightarrow \underline{x} \neq \underline{0}$ (rang C^T = rang $C = p$),

dvs. CAC^T er pos. def.

(specielt gælder CC^T pos. def., idet $CC^T = CIC^T$)

13 $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \Sigma \underline{v}$, Σ pos. def.

i $\left.\begin{array}{l} \langle \underline{u}, \underline{u} \rangle \geq 0 \text{ for alle } \underline{u} \\ \langle \underline{u}, \underline{u} \rangle = 0 \Leftrightarrow \underline{u} = \underline{0} \end{array}\right\}$ jf. def. af pos. def. matrix

ii $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \Sigma \underline{v} = (\underline{u}^T \Sigma \underline{v})^T = \underline{v}^T \Sigma \underline{u}$

$= \langle \underline{v}, \underline{u} \rangle$

iii $\langle \underline{u}, \underline{v} + \underline{w} \rangle = \underline{u}^T \Sigma (\underline{v} + \underline{w}) = \underline{u}^T \Sigma \underline{v} + \underline{u}^T \Sigma \underline{w}$

$= \langle \underline{u}, \underline{v} \rangle + \langle \underline{u}, \underline{w} \rangle$

iv $r\,\underline{u}^T \Sigma \underline{v} = (r\underline{u})^T \Sigma \underline{v} = \underline{u}^T \Sigma (r\underline{v})$

$\Rightarrow r \langle \underline{u}, \underline{v} \rangle = \langle r\underline{u}, \underline{v} \rangle = \langle \underline{u}, r\underline{v} \rangle$

$\Rightarrow \langle \underline{u}, \underline{v} \rangle$ definerer et indre produkt

14 Betragt $(\underline{Y} - \underline{\mu})^T \Sigma^{-1} (\underline{Y} - \underline{\mu})$

Sæt $\underline{Y} = \Sigma^{\frac{1}{2}} \underline{X} + \underline{\mu}$:

$(\underline{Y} - \underline{\mu})^T \Sigma^{-1} (\underline{Y} - \underline{\mu}) = (\Sigma^{\frac{1}{2}} \underline{X})^T \Sigma^{-1} \Sigma^{\frac{1}{2}} \underline{X}$

$= \underline{X}^T \Sigma^{\frac{1}{2}} \Sigma^{-1} \Sigma^{\frac{1}{2}} \underline{X}$, da $\Sigma^{\frac{1}{2}}$ sym.

$= \underline{X}^T \underline{X} = \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$,

da $X_i \sim N(0,1)$, $i = 1, \dots, n$, uafh.,

jf. note s. 3

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15 $Y_i \sim N(\mu, \sigma^2)$, $i = 1, \dots, n$, uafh.

$X_i = \frac{Y_i - \mu}{\sigma} \sim N(0,1)$, $i = 1, \dots, n$, uafh.

$\Rightarrow \underline{X} \sim N_n(\underline{0}, I)$

$\underline{U} = C\underline{X}$, C er ortogonal med første rækkevektor $[\frac{1}{\sqrt{n}} \dots \frac{1}{\sqrt{n}}]$

$\underline{U} \sim N_n(\underline{0}, CIC^T) = N_n(\underline{0}, I)$

$\Rightarrow U_i \sim N(0,1)$, $i = 1, \dots, n$, uafh.

$U_1 = [\frac{1}{\sqrt{n}} \dots \frac{1}{\sqrt{n}}]\underline{X} = \frac{1}{\sqrt{n}} \sum_i X_i = \sqrt{n}\,\bar{X} \Rightarrow \bar{X} = \frac{U_1}{\sqrt{n}}$

$$\frac{(n-1)S^2}{\sigma^2} = \sum_i \left(\frac{Y_i - \bar{Y}}{\sigma}\right)^2 = \sum_i \left(\frac{Y_i - \mu}{\sigma} - \frac{\bar{Y} - \mu}{\sigma}\right)^2$$

$$= \sum_i (X_i - \bar{X})^2 = \sum_i X_i^2 - n\bar{X}^2 = \sum_i U_i^2 - n\left(\frac{U_1}{\sqrt{n}}\right)^2$$

$$= U_2^2 + \dots + U_n^2 \sim \chi^2(n-1) \quad *$$

16 $$\mathrm{Var}(\underline{Y}_1, \underline{Z}) = \begin{bmatrix} I_{n_1} & 0 \\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{n_2} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_{n_1} & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I_{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix} \begin{bmatrix} I_{n_1} & -\Sigma_{11}^{-1}\Sigma_{12} \\ 0 & I_{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22\cdot 1} \end{bmatrix}$$

$\Rightarrow \underline{Y}_1$ og $\underline{Z}$ er uafh.

* Bemærk, at $\bar{Y} = \frac{\sigma U_1}{\sqrt{n}} + \mu$ og $S^2 = \frac{\sigma^2}{n-1}(U_2^2 + \dots + U_n^2)$, dvs. det ses også her, at $\bar{Y}$ og S^2 er uafhængige.