3.11

132 a

$$G(s) = 1 \qquad p_0 = 1$$

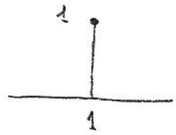
$$G^{(j)}(s) = 0 \qquad p_j = 0, \quad j = 1, 2, \ldots$$

b

$$G(s) = s \qquad p_0 = 0$$

$$G'(s) = 1 \qquad p_1 = 1$$

$$G^{(j)}(s) = 0 \qquad p_j = 0, \quad j = 2, 3, \ldots$$



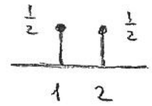
c

$$G(s) = \tfrac{1}{2}(s + s^2) \qquad p_0 = 0$$

$$G'(s) = \tfrac{1}{2} + s \qquad p_1 = \tfrac{1}{2}$$

$$G''(s) = 1 \qquad p_2 = \tfrac{1}{2}$$

$$G^{(j)}(s) = 0 \qquad p_j = 0, \quad j = 3, 4, \ldots$$



135 a

$$X \sim g(p)$$

$$G_X(s) = \sum_{k=1}^{\infty} s^k p(1-p)^{k-1} = ps \sum_{k=1}^{\infty} (s(1-p))^{k-1}$$

$$= ps \sum_{k=0}^{\infty} (s - ps)^k = ps \frac{1}{1-(s-ps)} = \frac{ps}{1-s+ps}$$

b

$$Y \sim nb(r, p), \quad Y = \sum_{j=1}^{r} X_j, \quad X_j \sim g(p), \quad j = 1, \ldots, r$$

uafh.

$$G_Y(s) = (G_X(s))^r = \left(\frac{ps}{1-s+ps}\right)^r$$

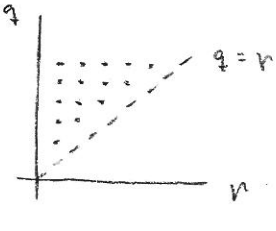
136

X, Y pos. heltallige, uafh., $p, q \in \mathbb{N}$, $q > p$

$$P\left(\frac{X}{X+Y} = \frac{p}{q}\right) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} P(X = p, X+Y = q)$$

$$P(X = p, X+Y = q) = P(X = p, Y = q-p) = P(X = p)\, P(Y = q-p)$$

$$\sum_{q=1}^{\infty} \sum_{p=1}^{q-1} s^{q-1} p\, P(X = p, X+Y = q) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} p s^{p-1} P(X = p)\, s^{q-p} P(Y = q-p)$$



$$= \sum_{p=1}^{\infty} p s^{p-1} P(X = p) \sum_{q=p+1}^{\infty} s^{q-p} P(Y = q-p)$$

$$= G_X'(s) \sum_{q=1}^{\infty} s^q P(Y = q)$$

$$= G_X'(s)\, G_Y(s)$$

fortsættes

<u>3.11</u>

136 fortsat

$$\int_0^1 G_X'(s)\, G_Y(s)\, ds = \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \left(\int_0^1 s^{q-1} ds \right) r \, P\left(\frac{X}{X+Y} = \frac{r}{q}\right)$$

$$= \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \left[\frac{s^q}{q} \right]_0^1 r \, P\left(\frac{X}{X+Y} = \frac{r}{q}\right)$$

$$= \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \frac{r}{q} P\left(\frac{X}{X+Y} = \frac{r}{q}\right) = E\left[\frac{X}{X+Y}\right]$$

altså $E\left[\frac{X}{X+Y}\right] = \int_0^1 G_X'(s)\, G_Y(s)\, ds$

137 $\left.\begin{array}{l} X_1, X_2, \dots \text{ ensfordelte ikke-neg. heltallige} \\ N \sim p(\lambda) \end{array}\right\}$ alle uafh.

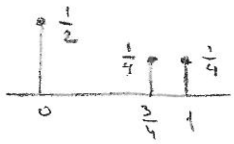
$$S_N = \sum_{k=1}^{N} X_k \sim p(\mu)$$

$$G_{S_N}(s) = (G_N \circ G_X)(s) = e^{\mu(s-1)} = e^{\lambda\left(\frac{\mu}{\lambda} s - \frac{1}{\lambda}\right)}$$

$$= e^{\lambda\left(1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda} s - 1\right)}$$

$$\Rightarrow G_X(s) = 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda} s \Rightarrow X \sim b\left(1, \frac{\mu}{\lambda}\right)$$

138 X :



$Y = 4X$ er heltallig

$N \sim p(20)$ $\quad X, N$ uafh.

$$E[4S_N] = EN\,EY = 20 \cdot \left(3\tfrac{1}{4} + 4\tfrac{1}{4}\right) = 20\tfrac{7}{4} = 35$$

$$ES_N = \tfrac{1}{4}\, 35 = 8{,}75 \$$$

140 $\left.\begin{array}{l} X_k \sim U\{1,2,3,4\}, \quad k = 1, \dots, N \\ N \sim p(4) \end{array}\right\}$ alle uafh.

$$Y = \sum_{k=1}^{N} X_k$$

a $\quad G_X(s) = s\tfrac{1}{4} + s^2\tfrac{1}{4} + s^3\tfrac{1}{4} + s^4\tfrac{1}{4} = \tfrac{1}{4}(s + s^2 + s^3 + s^4)$

$$G_N(s) = e^{4(s-1)}$$

$$G_Y(s) = (G_N \circ G_X)(s) = \exp(s + s^2 + s^3 + s^4 - 4)$$

fortsættes

3.11

140 fortsat

b $P(Y=0) = G_Y(0) = e^{-4}$

c $EY = EN \cdot EX = 4 \cdot \frac{5}{2} = 10$

$$\begin{aligned}
\operatorname{Var} Y &= EN \cdot \operatorname{Var} X + \operatorname{Var} N\,(EX)^2 \\
&= EN\,\left(E[X^2] - (EX)^2\right) + \operatorname{Var} N\,(EX)^2 \\
&= EN \cdot E[X^2] + (\operatorname{Var} N - EN)\,(EX)^2 \\
&= 4 \cdot \frac{4 \cdot 5 \cdot 9}{6} \cdot \frac{1}{4} + (4-4)\,(EX)^2 \\
&= 30
\end{aligned}$$

142 $X_1, X_2, \ldots$ ikke-neg. ensfordelte m. ford. fkt. F
N ikke-neg. heltallig m. sands. fr. br. fkt G } alle uafh.

$M_N = \text{maks}\{X_1, X_2, \ldots, X_N\}$, $M_N = 0$ for $N = 0$

a $$F_{M_N}(x) = P(M_N \le x) = \sum_{n=0}^{\infty} P(M_N \le x \mid N = n)\, P(N=n)$$
$$= \sum_{n=0}^{\infty} (F(x))^n\, P(N=n) = (G \circ F)(x)$$

M_N kont., når X kont.

b $X \sim U[0;100]$, dvs. $F(x) = \frac{x}{100}$, $0 \le x \le 100$

$N \sim p(2)$

$F_{M_N}(x) = e^{2\left(\frac{x}{100} - 1\right)} \ge 0{,}9 \Rightarrow x \ge 94{,}73$

147 $X_1, X_2, \ldots$ ensfordelte m. moment fr.br. fkt. M_X
N ikke-neg. heltallig m. sands. fr. br. fkt. G_N } alle uafh.

a $$M_{S_N}(t) = E\left[e^{tS_N}\right] = \sum_{n=0}^{\infty} E\left[e^{tS_N} \mid N = n\right] P(N=n)$$
$$= \sum_{n=0}^{\infty} E\left[e^{tS_n}\right] P(N=n) = \sum_{n=0}^{\infty} (M_X(t))^n\, P(N=n)$$
$$= (G_N \circ M_X)(t)$$

b $X_k \sim e(\lambda)$, $N \sim g(p)$

$$M_{S_N}(t) = \frac{p\,\frac{\lambda}{\lambda - t}}{1 - \frac{\lambda}{\lambda - t} + p\,\frac{\lambda}{\lambda - t}} = \frac{\lambda p}{\lambda p - t} \Rightarrow S_N \sim e(\lambda p)$$

fortsættes

3.11

147 b fortsat

S_N kan opfattes som ventetid mel. begivenheder i en udtyndet Poissonprocess (opg. 3.12 153)

c $E\,S_N = (G_N' \circ M_X)(0) \cdot M_X'(0) = G_N'(1) \cdot M_X'(0)$

$= EN \cdot EX$

149 $G(s,t) = E[s^X t^Y]$, X, Y ikke-neg. heltallige

$0 \le s,t \le 1$

a $G_X(s) = G(s,1)$, $G_Y(t) = G(1,t)$

b $EX = G_s(1,1)$, $EY = G_t(1,1)$

$\mathrm{Var}\,X = G_{s^2}(1,1) + G_s(1,1) - (G_s(1,1))^2$

$\mathrm{Var}\,Y = G_{t^2}(1,1) + G_t(1,1) - (G_t(1,1))^2$

$\mathrm{Cov}[X,Y] = G_{st}(1,1) - G_s(1,1)\,G_t(1,1)$

c i $G(s,t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s^j t^k p_{jk}$, $p_{00} = G(0,0)$.

$G_{s^j t^k}(0,0) = j!\,k!\,p_{jk} \Rightarrow p_{jk} = \dfrac{G_{s^j t^k}(0,0)}{j!\,k!}$

dvs. p_{jk} entydigt bestemt af G for alle (j,k)

ii X, Y uafh. $\Rightarrow$ $G(s,t) = E[s^X]\,E[t^Y]$

$= G_X(s)\,G_Y(t)$

$G(s,t) = G_X(s)\,G_Y(t) \Rightarrow G_{s^j t^k}(0,0) = G_X^{(j)}(0)\,G_Y^{(k)}(0)$

$\Rightarrow j!\,k!\,p_{jk} = j!\,p_j\,k!\,p_k \Rightarrow p_{jk} = p_j\,p_k$

$\Rightarrow$ X, Y uafh.

d i $M(s,t) = E[e^{sX} e^{tY}]$

ii $M_X(s) = M(s,0)$, $M_Y(t) = M(0,t)$

iii $EX = M_s(0,0)$, $\mathrm{Var}\,X = M_{s^2}(0,0) - (M_s(0,0))^2$

$EY = M_t(0,0)$, $\mathrm{Var}\,Y = M_{t^2}(0,0) - (M_t(0,0))^2$

$\mathrm{Cov}[X,Y] = M_{st}(0,0) - M_s(0,0)\,M_t(0,0)$

3.12

150 Poissonproces m. intensitet $\lambda = \frac{1}{13}$ pr. år

a $t = 10$, $X \sim p\left(\frac{10}{13}\right)$

$P(X=0) = e^{-\frac{10}{13}} = 0,4634$

b $T \sim e\left(\frac{1}{13}\right)$

$P(T > 5) = e^{-\frac{5}{13}} = 0,6807$

c $t = 13$, $Y \sim p\left(\frac{13}{13}\right) = p(1)$

$P(Y=1) = e^{-1} = 0,3679$

d $(P(X=1))^3 = \left(\frac{10}{13} e^{-\frac{10}{13}}\right)^3 = 0,0453$

151 Poissonproces m. intensitet $\lambda = 2$ pr. uge

a $t = \frac{1}{7}$, $X \sim p\left(\frac{2}{7}\right)$

$P(X=0) = e^{-\frac{2}{7}} = 0,7515$

b $P(\text{personskade}) = \frac{1}{10}$

udtyndet Poissonproces m. intensitet $\lambda p = 2 \frac{1}{10} = \frac{1}{5}$

$t = \frac{30,44}{7}$ (en gns. måned sat til 30,44 dg.)

$Y \sim p(\lambda p t) = p\left(\frac{30,44}{35}\right)$

$P(Y \geq 1) = 1 - P(Y=0) = 1 - e^{-\frac{30,44}{35}} = 0,5809$

c N: antal ulykkesfri uger pr. år , $Z \sim p(2)$

$N \sim b(52, P(Z=0)) = b(52, e^{-2})$

*
155 $X \sim p(2t)$, $Y \sim p(3t)$ uafh. , $X+Y \sim p(5t)$

a $t=1$, $P(X+Y=2) = \frac{e^{-5} 5^2}{2!} = 0,0842$

b $t=1$, $P(X=1, Y=1) = P(X=1)\,P(Y=1) = 2e^{-2}\,3e^{-3} = 0,0404$

c $t=1$, $P(Y=2 \mid X+Y=2) = \binom{2}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^0 = \left(\frac{3}{5}\right)^2 = \frac{9}{25} = 0,36$

d $t=10$, $Z_1, \ldots, Z_4 \sim U[0;10]$ uafh.

$\binom{4}{2} P(Z_j \leq 5, Z_k \leq 5) = \binom{4}{2} P(Z_j \leq 5)\, P(Z_k \leq 5)$, $j \neq k$

$= 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8} = 0,375$

* Løsning til opgave 154, se nederst næste side

3.12

156 Poissonproces m. intensitet $\lambda = 2$ pr. uge , $X \sim p(2)$

a $P(Y = k) = \dfrac{e^{-2}\, 2^{2k-1}}{(2k-1)!}$, $k = 1, 2, \ldots$

svarer ikke til en Poissonproces

b $P(X \le 1) = P(X=0) + P(X=1) = e^{-2} + 2e^{-2} = 3e^{-2} = 0{,}4060$

c $P(Z = k) = \dfrac{e^{-2}\, 2^{2k}}{(2k)!}$, $k = 0, 1, \ldots$

$$\sum_{k=0}^{\infty} P(Z=k) = e^{-2} \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} = e^{-2}\cosh 2 = \frac{1+e^{-4}}{2} = 0{,}5092$$

157 To Poissonprocesser m. intensitet hhv. μ og λ

X: antal begivenheder i første proces mellem to konsekutive begivenheder i anden proces

T: ventetid ml. to konsekutive begivenheder i anden proces.

$P(X = k(T)) = \int_0^{\infty} P(X=k)\, f_T(t)\, dt$, jf. sætn. 3.5.1 (b) side 170

$$= \int_0^{\infty} \frac{e^{-\mu t}(\mu t)^k}{k!} \lambda e^{-\lambda t}\, dt$$

$$= \frac{\mu^k \lambda}{(\mu+\lambda)^{k+1}} \int_0^{\infty} \frac{(\mu+\lambda)^{k+1}}{\Gamma(k+1)} t^{(k+1)-1} e^{-(\mu+\lambda)t}\, dt$$

$$= \left(\frac{\mu}{\mu+\lambda}\right)^k \frac{\lambda}{\mu+\lambda}\, 1 \quad , \quad k = 0, 1, \ldots$$

$$\Rightarrow \quad P(Y = j(T)) = \left(\frac{\mu}{\mu+\lambda}\right)^{j-1} \frac{\lambda}{\mu+\lambda} \quad , \quad j = 1, 2, \ldots$$

$$Y \sim g\left(\frac{\lambda}{\mu+\lambda}\right)$$

154 $P(X=k) = \binom{n}{k} \left(\dfrac{\lambda_1 t}{\lambda_1 t + \lambda_2 t}\right)^k \left(\dfrac{\lambda_2 t}{\lambda_1 t + \lambda_2 t}\right)^{n-k}$, $k = 0, 1, \ldots, n$