4.2

1 $X_1, X_2, \ldots$, $EX_k = \mu$, $\text{Var}\, X_k = \sigma^2$, $k = 1, 2, \ldots$

$$\text{Cov}(X_j, X_k) < 0 \quad , \quad j \neq k$$

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_i$$

$$\text{Var}\, \bar{X} = \frac{1}{n^2} \left(\sum_k \text{Var}\, X_k + \sum_{j \neq k} \text{Cov}(X_j, X_k) \right) < \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$P(|\bar{X} - \mu| > \varepsilon) \leq \frac{\text{Var}\, \bar{X}}{\varepsilon^2} < \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ for } n \to \infty$$

$$\Leftrightarrow \quad \bar{X} \xrightarrow{P} \mu$$

2 $X_1, X_2, \ldots$ $\quad X_n \xrightarrow{P} a$ dvs. $P(|X_n - a| > \varepsilon) \to 0$ for $n \to \infty$

$Y_1, Y_2, \ldots$ $\quad Y_n \xrightarrow{P} b$ dvs. $P(|Y_n - b| > \varepsilon) \to 0$ for $n \to \infty$

$$P(|X_n + Y_n - (a+b)| > \varepsilon) = P(|X_n - a + Y_n - b| > \varepsilon)$$

$$\leq P(|X_n - a| + |Y_n - b| > \varepsilon)$$

$$\leq P(|X_n - a| > \tfrac{\varepsilon}{2}) + P(|Y_n - b| > \tfrac{\varepsilon}{2}) \to 0$$

for $n \to \infty$

$$\Rightarrow X_n + Y_n \xrightarrow{P} a + b$$

3 $X_1, X_2, \ldots$ uafh. , $X_k \sim U[0;1]$, $k = 1, 2, \ldots$

$g : [0;1] \to \mathbb{R}$ reel fkt.

$$\frac{1}{n} \sum_{k=1}^{n} g(X_k) \xrightarrow{P} E[g(X_k)] = \int_0^1 g(x) \cdot 1 \, dx = \int_0^1 g(x)\, dx$$

4 $X_1, \ldots, X_n$ uafh., ensfordelte m. tath. $f(x) = 3x^2$, $0 \leq x \leq 1$

$$H_n = \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{X_k} \right)^{-1} \Leftrightarrow H_n^{-1} = \frac{1}{n} \sum_{k=1}^{n} Y_k \ , \ Y_k = \frac{1}{X_k}$$

$$y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f\left(\tfrac{1}{y}\right) \left| -\frac{1}{y^2} \right| = 3y^{-4} , \quad 1 \leq y < \infty$$

$$EY_k = \int_1^\infty y \cdot 3y^{-4}\, dx = \left[-\tfrac{3}{2} y^{-2} \right]_1^\infty = \frac{3}{2}$$

$$H_n^{-1} \xrightarrow{P} \frac{3}{2} \Rightarrow H_n \xrightarrow{P} \frac{2}{3}$$ jf. korollar 4.2.3 s. 273

4.2

5 $X_1, X_2, \ldots,$ uafh., ensfordelte m. tæth. $f(x) > 0,$ $0 \le x \le a$

ford.-fkt. F er voks. fkt. af x, $0 \le x \le a$

$$P(|X_{(1)} - 0| > \varepsilon) = P(X_{(1)} > \varepsilon) = (1 - F(\varepsilon))^n \to 0 \text{ for } n \to \infty$$

$$\Leftrightarrow \quad X_{(1)} \xrightarrow{P} 0$$

4.3

8 a $X \sim \Gamma(n, \lambda)$, $EX = \frac{n}{\lambda}$, $\operatorname{Var} X = \frac{n}{\lambda^2}$

$X = \sum_{k=1}^{n} Y_k$, $Y_k \sim e(\lambda)$, $k = 1, \ldots, n$, uafh.

$X \sim N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right)$ appr.

b $X \sim p(\lambda)$, λ 'stor', $EX = \lambda$, $\operatorname{Var} X = \lambda$

$X = \sum_{k=1}^{m} Y_k$, $Y_k \sim p\left(\frac{\lambda}{m}\right)$, $k = 1, \ldots, m$, uafh.

$X \sim N(\lambda, \lambda)$ appr.

11 $X \sim U\{1, \ldots, 6\}$, $EX = 3{,}5$, $\operatorname{Var} X = \frac{35}{12}$

$\bar{X} \sim N\left(3{,}5\,;\ \frac{35}{12n}\right)$ appr.

$$P(3 < \bar{X} < 4) = P(|\bar{X} - 3{,}5| < 0{,}5) = 2\,\Phi\left(\frac{0{,}5}{\sqrt{\frac{35}{12n}}}\right) - 1 > 0{,}99$$

$$\Rightarrow \Phi\left(\frac{0{,}5}{\sqrt{\frac{35}{12n}}}\right) > 0{,}995 \Rightarrow \frac{0{,}5}{\sqrt{\frac{35}{12n}}} > 2{,}576 \Rightarrow n > 77{,}42$$

$n \ge 78$

13 multiple choice, 100 spm. m. hv. 4 svarmuligheder

$P(\text{rigtigt svar}) = 1 \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{13}{16}$ $\left(P(\text{viden}) = \frac{3}{4}\right)$

X : antal rigtige svar, $X \sim b\left(100\,\frac{13}{16},\ 100\,\frac{13}{16}\,\frac{3}{16}\right)$

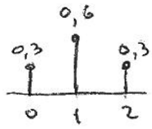
$X \sim N\left(\frac{325}{4}, \frac{975}{64}\right)$ appr.

$$P(\text{bestå}) = P(X \ge 80) \approx \Phi\left(-\frac{79{,}5 - 81{,}25}{\frac{5}{8}\sqrt{39}}\right) = \Phi(0{,}44836)$$

$$= 0{,}6857 \quad (\text{kontinuitetskorrektion anvendt})$$

4.3

15 200 lejl. , antal biler pr. lejl.



$\mu = 0{,}6 + 2 \cdot 0{,}3 = 1{,}2$

$\sigma^2 = 0{,}6 + 4 \cdot 0{,}3 - (1{,}2)^2$
$= 1{,}8 - 1{,}44 = 0{,}36$

X : antal biler

$X \sim N(200 \cdot 1{,}2 \;,\; 200 \cdot 0{,}36) = N(240 \,,\, 72)$ app.

$\Phi\left(\frac{x-240}{\sqrt{72}}\right) > 0{,}95 \Rightarrow \frac{x-240}{\sqrt{72}} > 1{,}645 \Rightarrow x > 253{,}96$

$x \geq 254$

16 i $EX = \mu$, $\operatorname{Var} X = \sigma^2$, g diff. fkt.

$$g(X) = g(\mu) + g'(\mu)(X-\mu) + \begin{cases} R_1 \\ \frac{1}{2} g''(\mu)(X-\mu)^2 + R_2 \end{cases}$$

$$E[g(X)] \approx \begin{cases} g(\mu) + 0 = g(\mu) \\ g(\mu) + 0 + \frac{1}{2} g''(\mu)\sigma^2 = g(\mu) + \frac{1}{2}\sigma^2 g''(\mu) \end{cases}$$

ii $X \sim U[0;1]$, $\mu = \frac{1}{2}$, $\sigma^2 = \frac{1}{12}$

a $g(x) = x$, $EX \approx \begin{cases} \frac{1}{2} \\ \frac{1}{2} + 0 = \frac{1}{2} \end{cases}$

b $g(x) = x^2$, $EX \approx \begin{cases} \frac{1}{4} \\ \frac{1}{4} + \frac{1}{2}\,\frac{1}{12} \cdot 2 = \frac{3+1}{12} = \frac{1}{3} \end{cases}$

c $g(x) = x^k$, $k = 3, 4, \ldots$ $EX \approx \begin{cases} (\frac{1}{2})^k \\ (\frac{1}{2})^k + \frac{1}{2}\,\frac{1}{12}\,k(k-1)(\frac{1}{2})^{k-2} \end{cases}$
$= \frac{12 + 2k(k-1)}{12} (\frac{1}{2})^k$

d $g(x) = e^x$, $EX \approx \begin{cases} e^{\frac{1}{2}} = 1{,}6487 \\ e^{\frac{1}{2}} + \frac{1}{2}\,\frac{1}{12} e^{\frac{1}{2}} = \frac{25}{24} e^{\frac{1}{2}} = 1{,}7174 \end{cases}$

e $g(x) = e^{-x}$, $EX \approx \begin{cases} e^{-\frac{1}{2}} = 0{,}6065 \\ e^{-\frac{1}{2}} + \frac{1}{2}\,\frac{1}{12} e^{-\frac{1}{2}} = \frac{25}{24} e^{-\frac{1}{2}} = 0{,}6318 \end{cases}$

f $g(x) = \sin(\pi x)$, $EX \approx \begin{cases} \sin\frac{\pi}{2} = 1 \\ \sin\frac{\pi}{2} + \frac{1}{2}\,\frac{1}{12}\pi^2 \sin\frac{\pi}{2} = 1 - \frac{\pi^2}{24} \end{cases}$
$= 0{,}5888$

g $g(x) = \sin(2\pi x)$, $EX \approx \begin{cases} \sin \pi = 0 \\ \sin\pi + \frac{1}{2}\,\frac{1}{12}\pi^2 \sin\pi = 0 + 0 \end{cases}$
$= 0$

4.3

18 opg. 4.2 4 fortsat

$$EY_k^2 = \int_1^\infty y^2 \cdot 3y^{-4}\,dy = \left[-3y^{-1}\right]_1^\infty = 3$$

$$\operatorname{Var} Y_k = 3 - \left(\tfrac{3}{2}\right)^2 = \tfrac{3}{4}$$

$$\bar{Y} \sim N\left(\tfrac{3}{2}, \tfrac{3}{4}\right) \text{ app.}$$

$$H_n = \bar{Y}^{-1} \sim N\left(\left(\tfrac{3}{2}\right)^{-1}, \frac{\frac{3}{4}\left(-\frac{1}{(\frac{3}{2})^2}\right)}{n}\right) = N\left(\tfrac{2}{3}, \tfrac{4}{27n}\right) \text{ app.}$$

jf. sætn. 4.3.2 s. 280

4.4

19 $X \sim h(N, r, n)$

$$P(X = k) = \frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}} = \frac{(r)_k\,(N-r)_{n-k}\;n!}{k!\,(n-k)!\,(N)_n}$$

$$= \frac{\frac{(N-r)_n}{(N-k)_n}\;\frac{(N-n)_k}{(N-r-n+k)_k}\;\frac{(n)_k\,r^k}{(N)_k}\;\frac{(r)_k}{r^k}}{k!} \rightarrow \frac{e^{-\lambda}\cdot 1\cdot \lambda^k\cdot 1}{k!} \quad \text{for} \begin{cases} r \to \infty \\ n \to \infty \\ N \to \infty \\ \frac{r}{N} \to 0 \\ \frac{n}{N} \to 0 \\ \frac{rn}{N} = \lambda \ (\text{konst.}) \end{cases}$$

dvs. $X \xrightarrow{D} Y \sim p(\lambda)$

Desuden

$$P(X = k) = \binom{n}{k}\frac{(r)_k}{(N)_k}\frac{(N-r)_{n-k}}{(N-k)_{n-k}} \rightarrow \binom{n}{k}p^k(1-p)^{n-k}$$

$$\text{for} \begin{cases} r \to \infty \\ N \to \infty \\ \frac{r}{N} \to p \ (\text{konst.}) \end{cases}$$

dvs. $X \xrightarrow{D} Y \sim b(n, p)$

20 $X_1, X_2, \dots, X_n \sim U\{1, \dots, 6\}$; $X \sim U[0;1]$

$$P\left(\tfrac{1}{n}X_n \le x\right) = P(X_n \le nx) = \frac{[nx]}{n} \rightarrow x \text{ for } n \to \infty, \quad 0 \le x \le 1$$

$$\Leftrightarrow F_{\frac{1}{n}X_n}(x) \rightarrow F_X(x) \text{ for } n \to \infty$$

$$\Leftrightarrow \tfrac{1}{n}X_n \xrightarrow{D} X$$

4.4

21 $X_1, X_2, \dots, X_n \sim g(\frac{1}{n})$; $X \sim e(1)$

$$P(\tfrac{1}{n}X_n \le x) = P(X_n \le nx) = \sum_{k=1}^{[nx]} \tfrac{1}{n}(1-\tfrac{1}{n})^{k-1}$$

$$= \tfrac{1}{n}(1-\tfrac{1}{n})\frac{1-(1-\frac{1}{n})^{[nx]}}{1-(1-\frac{1}{n})} = (1-\tfrac{1}{n})(1-(1-\tfrac{1}{n})^{[nx]}), \quad [xn]=m, \ \tfrac{1}{n} \approx \tfrac{x}{m}$$

$$\approx (1-\tfrac{x}{m})(1-(1-\tfrac{x}{m})^m) \to 1-e^{-x} \text{ for } m \to \infty, \ 0 \le x < \infty$$

$$\Leftrightarrow F_{\frac{1}{n}X_n}(x) \to F_X(x) \text{ for } n \to \infty \Leftrightarrow \tfrac{1}{n}X_n \xrightarrow{D} X$$

22 $X_1, X_2, \dots$ uafh., $X_n \sim U[0;1]$

$$F_{X_{(n)}}(x) = (F_{X_k}(x))^n = x^n \ ; \ Y_n = n(1-X_{(n)}), \ Y \sim e(1)$$

$$P(Y_n \le y) = P(n(1-X_{(n)}) \le y) = P(X_{(n)} \ge 1-\tfrac{y}{n})$$

$$= 1-(1-\tfrac{y}{n})^n \to 1-e^{-y} \text{ for } n \to \infty, \ 0 \le y < \infty$$

$$\Leftrightarrow F_{Y_n}(y) \to F_Y(y) \text{ for } n \to \infty \Leftrightarrow Y_n \xrightarrow{D} Y$$

24 $X_1, X_2, \dots$ uafh., ensfordelte m. tæth. $f(x) = 2x$, $0 \le x \le 1$

$$F_{X_k}(x) = \int_0^1 2x\,dx = x^2, \ 0 \le x \le 1$$

$$Y_n = \sqrt{n}\,X_{(1)}, \quad F_{X_{(1)}}(x) = 1-(1-x^2)^n, \ 0 \le x \le 1$$

$$F_{Y_n}(y) = P(Y_n \le y) = P(\sqrt{n}X_{(1)} \le y) = P(X_{(1)} \le \tfrac{y}{\sqrt{n}})$$

$$= 1-(1-(\tfrac{y}{\sqrt{n}})^2)^n = 1-(1-\tfrac{y^2}{n})^n \to 1-e^{-y^2} \text{ for } n \to \infty, \ 0 \le y < \infty$$

$$\Leftrightarrow Y_n \xrightarrow{D} Y \sim W(2,1)$$

25 $X_1, X_2, \dots$ uafh., ensfordelte m. tæth. $f(x) = 3x^2$, $0 \le x \le 1$

$$F_{X_k}(x) = \int_0^1 3x^2\,dx = x^3, \ 0 \le x \le 1$$

$$Y_n = a_n X_{(1)}, \quad F_{X_{(1)}}(x) = 1-(1-x^3)^n, \ 0 \le x \le 1$$

$$F_{Y_n}(y) = P(Y_n \le y) = P(a_n X_{(1)} \le y) = P(X_{(1)} \le \tfrac{y}{a_n})$$

$$= 1-(1-(\tfrac{y}{a_n})^3)^n = 1-(1-\tfrac{y^3}{a_n^3})^n$$

$$a_n = \sqrt[3]{n} \Rightarrow F_{Y_n}(y) = 1-(1-\tfrac{y^3}{n})^n \to 1-e^{-y^3} \text{ for } n \to \infty, \ 0 \le y < \infty$$

$$\Leftrightarrow Y_n \xrightarrow{D} Y \sim W(3,1)$$