5.3

1 Jf. opg. 2.4 24 a

$$X = \begin{cases} 0 \,, & \text{når } 0 \le u \le \frac{1}{8} \\ 1 \,, & \text{når } \frac{1}{8} < u \le \frac{1}{2} \\ 2 \,, & \text{når } \frac{1}{2} < u \le \frac{7}{8} \\ 3 \,, & \text{når } \frac{7}{8} < u \le 1 \end{cases} \,, \quad U \sim U[0;1]$$

2 Jf. opg. 2.4 24 b

$$X = \begin{cases} 0 \,, & \text{når } 0 \le u \le \frac{1}{4} \\ 1 \,, & \text{når } \frac{1}{4} < u \le \frac{3}{4} \\ 2 \,, & \text{når } \frac{3}{4} < u \le 1 \end{cases} \,, \quad U \sim U[0;1]$$

3 $Y \sim p(\lambda) = p(1 \cdot \lambda)$, dvs. Y kan opfattes som antal begivenheder i $[0;\lambda]$ i en poissonproces m. intensitet 1.

S_n: tid til n'te begivenhed

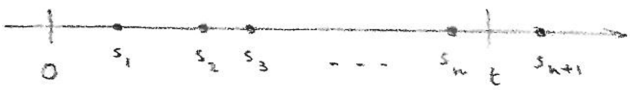
$S_n = \sum_{k=1}^{n} X_k \,, \quad X_k \sim e(1) \,, \; k = 1, \ldots, n \,,$ uafh.

$S_n = -\sum_{k=1}^{n} \ln U_k \,, \quad U_k \sim U[0;1] \,, \; k = 1, \ldots, n \,,$ uafh.

$$Y = \min\{n \mid S_{n+1} > \lambda\} = \min\{n \mid S_n > \lambda\} - 1$$
$$= \min\{n \mid -\sum_{k=1}^{n} \ln U_k > \lambda\} - 1$$

4 $N \sim p(\lambda t)$

$N = \min\{n \mid -\sum_{k=1}^{n} \ln U_k > \lambda t\} - 1$, Jf. opg. 3 med $\lambda := \lambda t$



$S_j = t\,U_{(j)} \,, \; j = 1, \ldots, n \,, \quad U_j \sim U[0;1] \,, \; j = 1, \ldots, n \,,$ uafh.

jf. 'order statistic property'

5 $X \sim nb(r, p)$

$X = \sum_{j=1}^{r} Y_j \,, \quad Y_j \sim g(p) \,, \; j = 1, \ldots, r \,,$ uafh.

$$P(Y_j = k) = P(Y_j \le k) - P(Y_j \le k-1) = P(Y_j > k-1) - P(Y_j > k)$$

$Y_j = k$, når $(1-p)^k < u < (1-p)^{k-1} \,, \quad U \sim U[0;1]$

Bemærk, at $(1-p)^k < u < (1-p)^{k-1} \iff \frac{\ln u}{\ln(1-p)} < k < \frac{\ln u}{\ln(1-p)} + 1$,

dvs. $Y_j = \left[\frac{\ln U}{\ln(1-p)}\right] + 1 \,, \quad U \sim U[0;1]$.

$$X = \sum_{j=1}^{r} \left[\frac{\ln U_j}{\ln(1-p)}\right] + r \,, \quad U_j \sim U[0;1] \,, \; j = 1, \ldots, r$$

uafh.

7 $\quad X$ har tæth. $f(x) = 3x^2 \,, \quad 0 \le x \le 1$

$$F(x) = \int_0^x 3t^2\,dt = \left[t^3\right]_0^x = x^3 \,, \quad 0 \le x \le 1$$

$$u = x^3 \iff x = u^{\frac{1}{3}}$$

$$X = U^{\frac{1}{3}} \,, \quad U \sim U[0;1]$$

8 $\quad X \sim$ Cauchy, tæth. $f(x) = \dfrac{1}{\pi(1+x^2)} \,, \quad -\infty < x < \infty$

a

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}\,dt = \frac{1}{\pi}\left[\operatorname{arctg} t\right]_{-\infty}^{x} = \frac{1}{\pi}\operatorname{arctg} x + \frac{1}{2}$$

$$u = \operatorname{arctg} x + \frac{1}{2} \iff x = \operatorname{tg}\left(\pi\left(u - \frac{1}{2}\right)\right)$$

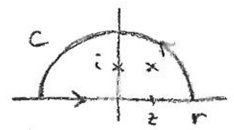
$$X = \operatorname{tg}\left(\pi\left(U - \frac{1}{2}\right)\right), \quad U \sim U[0;1]$$

b $\quad X, Y \sim$ Cauchy uafh.

$$Z = X + Y$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \, \frac{1}{\pi(1+(z-x)^2)}\,dx$$

$$= \frac{1}{\pi^2} \lim_{r\to\infty} \oint_C \frac{1}{1+w^2}\,\frac{1}{1+(z-w)^2}\,dw$$



singulære punkter: $w = \pm i$ og $w = z \pm i$

$$= \frac{1}{\pi^2}\, 2\pi i \left(\frac{1}{2i}\,\frac{1}{1+(z-i)^2} + \frac{1}{1+(z+i)^2}\,\frac{1}{2i} \right)$$

$$= \frac{1}{\pi}\left(\frac{1}{z^2 - 2iz} + \frac{1}{z^2 + 2iz} \right)$$

$$= \frac{1}{\pi z}\,\frac{z + 2i + z - 2i}{z^2 + 4} = \frac{2}{\pi(4 + z^2)}$$

$$f_{\frac{X+Y}{2}}(z) = \frac{2}{\pi(4 + (2z)^2)}\,|2| = \frac{1}{\pi(1+z^2)}$$

dvs. $\dfrac{X+Y}{2} \sim$ Cauchy

Heraf ses, at med $X_1, \ldots, X_n \sim$ Cauchy uafh., bliver $\overline{X} \sim$ Cauchy $(n = 2^m)$.

I Cauchy-fordelingen giver det altså ikke forøget nøjagtighed at tage gennemsnit af flere observationer. ($\overline{X} \sim$ Cauchy, $n \neq 2^m$, vises ved benyttelse af karakteristisk fkt.).

5.4

9 $X \sim W(\alpha, \lambda)$

$F(x) = 1 - e^{-\lambda x^\alpha}, \quad 0 \le x < \infty$

$u = 1 - e^{-\lambda x^\alpha} \Leftrightarrow x^\alpha = -\frac{1}{\lambda} \ln(1-u) \Leftrightarrow x = \left(-\frac{1}{\lambda} \ln(1-u)\right)^{\frac{1}{\alpha}}$

$X = \left(-\frac{1}{\lambda} \ln U\right)^{\frac{1}{\alpha}}, \quad U \sim U[0;1] \qquad (U \sim 1-U)$

10 $X \sim g(p)$

Fra opg. 2.6 66 : $X = [T] + 1, \quad T \sim e(\lambda), \quad p = 1 - e^{-\lambda}$

$T = -\frac{1}{\lambda} \ln U = \frac{\ln U}{\ln e^{-\lambda}} = \frac{\ln U}{\ln(1-p)}, \quad U \sim U[0;1]$

$X = \left[\frac{\ln U}{\ln(1-p)} \right] + 1$

Samme resultat som mellemresultatet i opg. 5.3 5.

11 $P(X = 0) = 0{,}2$

$F_X(x) = 1 - 0{,}8\, e^{-x}, \quad 0 \le x < \infty$

$u = 1 - 0{,}8\, e^{-x} \Leftrightarrow x = -\ln \frac{5(1-u)}{4}$

$X = \begin{cases} 0, & \text{når } 0{,}8 < u \le 1 \\ -\ln \frac{5U}{4}, & \text{når } 0 < u \le 0{,}8 \end{cases}, \quad U \sim U[0;1]$

13 $X \sim e(1), \quad X = -\ln U, \quad U \sim U[0;1]$

$\ln X = \ln(-\ln U), \quad U \sim U[0;1]$

$E[\ln X] \approx \frac{1}{n} \sum_{k=1}^{n} \ln(-\ln u_k), \quad U_k \sim U[0,1],$
$k = 1, \dots, n$, uafh.

14 $X \sim N(0,1)$, tæth. $\varphi(x)$, $Y = \sin X$

$\int_{-\infty}^{\infty} |\sin x|\, \varphi(x)\, dx \le \int_{-\infty}^{\infty} 1\, \varphi(x)\, dx = 1 \Rightarrow E[\sin X]$ eks.

$E[\sin X] = \int_{-\infty}^{\infty} \sin x\, \varphi(x)\, dx = 0$

simulering ikke nødvendig

15 $X, Y, Z \sim U[0;1]$ uafh.,

$P(X+Y+Z \le 2{,}5) \approx \frac{1}{n} \sum_{k=1}^{n} I_k, \quad I_k = \begin{cases} 1, & \text{når } U_1 + U_2 + U_3 \le 2{,}5 \\ 0, & \text{når } U_1 + U_2 + U_3 > 2{,}5 \end{cases},$
$U_1, U_2, U_3 \sim U[0;1]$ uafh.

5.4

16 $g: [0;1] \to R \,, \qquad I = \int_0^1 g(x)\,dx$

$$E[g(U)] = \int_0^1 g(u)\cdot 1\,du = I \,, \quad U \sim U[0;1]$$

$$I \approx \frac{1}{n}\sum_{k=1}^{n} g(u_k) \,, \quad U_k \sim U[0;1] \,,\ k = 1,\dots,n \,,\ \text{uafh.}$$

17 Box-Muller:

$$X = \sqrt{-2\ln U}\,\cos(2\pi V)$$
$$Y = \sqrt{-2\ln U}\,\sin(2\pi V) \,, \quad U, V \sim U[0;1] \ \text{uafh.}$$

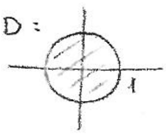
Transformation: $X_1 = \sigma_1 X + \mu_1$

$$Y_1|x_1 = \sigma_2\sqrt{1-\rho^2}\,Y + \mu_{Y|x_1}$$
$$= \sigma_2\sqrt{1-\rho^2}\,Y + \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1)$$

$$(X_1, Y_1) = \left(\sigma_1 X + \mu_1 \,,\ \sigma_2\left(\rho X + \sqrt{1-\rho^2}\,Y\right) + \mu_2\right)$$

$$(X_1, Y_1) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

18 D:



$$f(r,\theta) = \frac{1}{\pi} \,, \quad 0 \le r \le 1 \,,\ 0 \le \theta < 2\pi \,, \quad x = r\cos\theta \,,\ y = r\sin\theta$$

$$f_R(r) = \int_0^{2\pi} \frac{1}{\pi}\, r\, d\theta = \frac{r}{\pi}[\theta]_0^{2\pi} = 2r$$

$$F_R(r) = \int_0^r 2t\,dt = 2\left[\frac{t^2}{2}\right]_0^r = r^2 \,, \quad u = r^2 \Leftrightarrow r = \sqrt{u}$$

$$f_\Theta(\theta) = \int_0^1 \frac{1}{\pi}\, r\,dr = \frac{1}{\pi}\left[\frac{r^2}{2}\right]_0^1 = \frac{1}{2\pi} \Rightarrow \Theta \sim U[0;2\pi]$$

$$(X,Y) = \left(\sqrt{U}\cos(2\pi V) \,,\ \sqrt{U}\sin(2\pi V)\right) ,$$
$$U, V \sim U[0;1] \ \text{uafh}$$

$$(X,Y) \sim U[D]$$

19 X, tæth. $f(x) = \frac{4}{\sqrt{\pi}}\, x^2 e^{-x^2} \,, \quad 0 \le x < \infty$

$$Y \sim \Gamma(3,1) \,, \quad g(y) = \frac{1}{2} y^2 e^{-y}$$

$$\frac{f(y)}{c\,g(y)} = \frac{\frac{4}{\sqrt{\pi}} y^2 e^{-y^2}}{c\,\frac{1}{2} y^2 e^{-y}} = \frac{1}{c}\,\frac{8}{\sqrt{\pi}}\, e^{-(y^2 - y + \frac{1}{4})}\, e^{\frac{1}{4}}$$

$$= \frac{1}{c}\,\frac{8e^{\frac{1}{4}}}{\sqrt{\pi}}\, e^{-(y-\frac{1}{2})^2} \le 1 \quad \text{for } c = \frac{8e^{\frac{1}{4}}}{\sqrt{\pi}} \approx 5{,}80$$

fortsættes

5.4

19 fortsat

$$Y = \sum_{k=1}^{3} Z_k , \quad Z_k \sim e(1) , \quad Z_k = -\ln V_k , \quad k = 1, 2, 3$$

uafh.

$$\left.\begin{array}{l} Y = -\sum_{k=1}^{3} \ln V_k \\ U \sim U[0,1] \end{array}\right\} \text{ uafh.}$$

$X = y$, når $u < e^{-(y-\frac{1}{2})^2}$,

ellers afvis