7.2

4 c $$P^T - I = \begin{bmatrix} -0{,}2 & 0{,}6 & 0 \\ 0{,}1 & -0{,}8 & 0{,}6 \\ 0{,}1 & 0{,}2 & -0{,}6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -3{,}6 \\ 0 & 1 & -1{,}2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\pi} = r\,(3{,}6;\ 1{,}2;\ 1)\ ,\quad \sum_i \pi_i = 1 \Rightarrow r = \frac{1}{5{,}8} = \frac{5}{29}$$

$$\underline{\pi} = \left(\frac{18}{29}, \frac{6}{29}, \frac{5}{29} \right)$$

$$E_H[\tau_H] = \frac{1}{\pi_H} = \frac{29}{5} = 5{,}8$$

5 Fra eks. 7.2.4 s. 411 - 412:

Markovkæde $\{X_n\}$, $S_X = \{0,1\}$, $P_X = [p_{ij}] = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

Sæt $Y_n = (X_n, X_{n+1})$, $n = 0,1,\dots$, $S_Y = \{0,1\}^2$

$$P(Y_{n+1} = (j,k) \mid Y_0 = (i_0,i_1), Y_1 = (i_1,i_2), \dots, Y_{n-1} = (i_{n-1}, i), Y_n = (i,j))$$
$$= P(X_{n+1} = j, X_{n+2} = k \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = x_{n-1}, X_n = i, X_{n+1} = j)$$
$$= P(X_{n+1} = j, X_{n+2} = k \mid X_n = i, X_{n+1} = j)$$
$$= P(Y_{n+1} = (j,k) \mid Y_n = (i,j))$$

dvs. $\{Y_n\}$ opfylder Markovbetingelsen

Desuden ses, at $p_{(i,j),(j,k)} = p_{jk}$,

og $p_{(i,j),(m,k)} = 0$ for $m \neq j$

$$P_Y = \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \end{bmatrix}$$ (rækkefølge: (0,0), (0,1), (1,0), (1,1))

$$P_Y^T - I = \begin{bmatrix} -p & 0 & 1-p & 0 \\ p & -1 & p & 0 \\ 0 & q & -1 & q \\ 0 & 1-q & 0 & -q \end{bmatrix} \sim \cdots \sim \begin{bmatrix} p & 0 & -(1-p) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1-q & -q \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\pi} = r_1\left(\frac{q}{p}(1-p), q, q, 1-q \right) = r\,(q(1-p), pq, pq, p(1-q))$$

$$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{p+q} \Rightarrow \underline{\pi} = \left(\frac{q(1-p)}{p+q}, \frac{pq}{p+q}, \frac{pq}{p+q}, \frac{p(1-q)}{p+q} \right)$$

7.2

8 Irreducibel Markovkæde, S endelig, $\forall i,j \in S: p_{ij} = p_{ji}$,

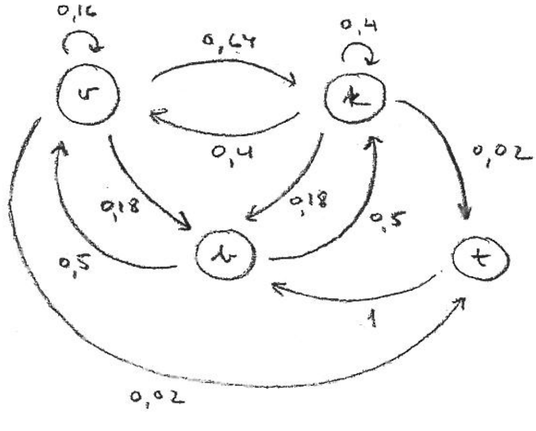
P kaldes dobbelt stokastisk, da $\sum_{j=1}^{m} p_{ij} = 1$ og $\sum_{i=1}^{m} p_{ij} = \sum_{i=1}^{m} p_{ji} = 1$

$(P^T - I)\,\underline{\pi}^T = \underline{0} \Leftrightarrow (P - I)\,\underline{\pi}^T = \underline{0}$ har løsning $\underline{\pi} = r\,\underline{1}_m$,

da $(P - I)\,\underline{1}_m = \left(\sum_{j=1}^{m} p_{ij} - 1, \dots, \sum_{j=1}^{m} p_{mj} - 1\right) = \underline{0}$

$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{m} \Rightarrow \underline{\pi} = \left(\frac{1}{m}, \dots, \frac{1}{m}\right)$

11



fortsættes

7.2

11 fortsat

a
$$P = \begin{bmatrix} 0{,}16 & 0{,}64 & 0{,}18 & 0{,}02 \\ 0{,}4 & 0{,}4 & 0{,}18 & 0{,}02 \\ 0{,}5 & 0{,}5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -0{,}84 & 0{,}4 & 0{,}5 & 0 \\ 0{,}64 & -0{,}6 & 0{,}5 & 0 \\ 0{,}18 & 0{,}18 & -1 & 1 \\ 0{,}02 & 0{,}02 & 0 & -1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1{,}48 & 1 & 0 & 0 \\ -0{,}496 & 0 & 1 & 0 \\ -0{,}0496 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\pi} = r\,(1\,;\ 1{,}48\,;\ 0{,}496\,;\ 0{,}0496)$$

$$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{3{,}0256}$$

$$\underline{\pi} = (0{,}3305\,;\ 0{,}4892\,;\ 0{,}1639\,;\ 0{,}01639)$$

b
$$E_t[N_b] = \frac{\pi_b}{\pi_t} = \frac{0{,}496}{0{,}0496} = 10$$

c Der er ikke taget højde for inddeling i afsnit.

12 Betjeningssted (én kunde ad gangen) med plads til tre i en kø.

Begivenhed: Afslutning af betjening eller ankomst af kunde(r).

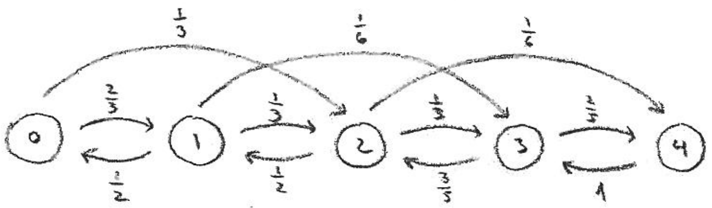
Ankomster: Sands. $\frac{2}{3}$ for aleneankomst.

Sands. $\frac{1}{3}$ for parvis ankomst. Hvis ikke der er plads til begge, går begge (tæller i så fald ikke med som begivenhed).

X_n : Antal kunder i kø eller under betjening umiddelbart efter en begivenhed.

$S = \{0, 1, 2, 3, 4\}$

a



$$\frac{r_{34}}{r_{32}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} = \frac{\frac{2}{5}}{\frac{3}{5}}$$

fortsættes

7.2

12 a fortsat

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{2}{3} & -1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{3}{5} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & -1 & 1 \\ 0 & 0 & \frac{1}{6} & \frac{2}{5} & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 1 & -48 & 84 \\ 0 & 0 & 0 & 14 & -25 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\pi} = r\,(9, 18, 24, 25, 14), \quad \sum_i \pi_i = 1 \Rightarrow r = \frac{1}{90}$$

$$\underline{\pi} = \left(\frac{1}{10}, \frac{1}{5}, \frac{4}{15}, \frac{5}{18}, \frac{7}{45}\right)$$

b $P(\text{ing. kunder}) = \pi_0 = \frac{1}{10} = 0{,}1$

$P(\text{køen fuld}) = \pi_4 = \frac{7}{45} = 0{,}1556$

c $$E_0[N_4] = \frac{\frac{7}{45}}{\frac{1}{10}} = \frac{70}{45} = \frac{14}{9} = 1{,}556$$

7.2

13 Grænsefordeling $\underline{q} = (q_1, \dots, q_r)$ for $\{X_n\}$

$$\forall j \in S: \quad P(X_{n+1} = j) = \sum_{i=1}^{r} P(X_{n+1} = j \mid X_n = i)\, P(X_n = i)$$

$$= \sum_{i=1}^{r} p_{ij}\, P(X_n = i)$$

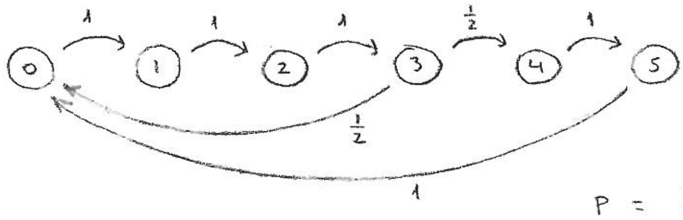
$$\Rightarrow \lim_{n \to \infty} P(X_{n+1} = j) = \lim_{n \to \infty} \sum_{i=1}^{r} p_{ij}\, P(X_n = i)$$

$$= \sum_{i=1}^{r} p_{ij} \lim_{n \to \infty} P(X_n = i)$$

$$\Leftrightarrow \quad q_j = \sum_{i=1}^{r} q_i\, p_{ij}$$

$\Leftrightarrow \quad \underline{q} = P\,\underline{q} \quad \Leftrightarrow \quad \underline{q}$ er stationær fordeling

14



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a 4

b sfd $\{4, 6\} = 2$

c
$$P^T - I = \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

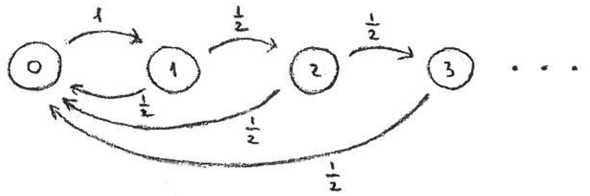
$\underline{\pi} = r\,(2, 2, 2, 2, 1, 1)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{10}$

$\underline{\pi} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10}\right)$, ikke grænsefordeling

15 $\pi_k = \frac{1}{2^{k+1}}$, $k = 0, 1, \dots$, jf. eks. 7.2.16 s. 427

a $E_{10}[\tau_{10}] = \frac{1}{\pi_{10}} = 2^{11} = 2048$

b $E_{10}[N_9] = \frac{\pi_9}{\pi_{10}} = \frac{2^{11}}{2^{10}} = 2$

16 

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Kæden er irreducibel

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & -1 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -\frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & -\frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\pi_1 = \pi_0 , \quad \pi_2 = \tfrac{1}{2}\pi_1 = \tfrac{1}{2}\pi_0 , \quad \pi_k = \tfrac{1}{2}\pi_{k-1} = \frac{1}{2^{k-1}}\pi_0 , \quad k = 1, 2, \ldots$$

$$\sum_{k=0}^{\infty} \pi_k = 1 \Rightarrow \pi_0 + \pi_0 \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 \Rightarrow \pi_0 \left(1 + \sum_{k=0}^{\infty} \frac{1}{2^k} \right) = 1$$

$$\Rightarrow \pi_0 (1 + 2) = 1 \Rightarrow \pi_0 = \frac{1}{3}$$

$$\pi_0 = \frac{1}{3} , \quad \pi_k = \frac{1}{3} \, \frac{1}{2^{k-1}} , \quad k = 1, 2, \ldots$$

Kæden er pos. rekurrent

$d(0) = \text{sfd}\,\{2, 3, \ldots\} = 1$

Kæden er aperiodisk

Sammenfattende: Kæden er ergodisk
Grænsefordelingen eksisterer, $\underline{\mathbf{q}} = \underline{\boldsymbol{\pi}}$

17 $\{X_n\}$ ergodisk Markovkæde, n 'stor'

Betragt sekvensen af besøgte tilstande i omvendt tid, dvs. en kæde med overgangssandsynligheder $q_{ij} = P(X_{n-1} = j \mid X_n = i)$.

a
$$\begin{aligned} q_{ij} &= P(X_{n-1} = j \mid X_n = i) \\ &= \frac{P(X_{n-1} = j, X_n = i)}{P(X_n = i)} \\ &= \frac{P(X_n = i \mid X_{n-1} = j)\, P(X_{n-1} = j)}{P(X_n = i)} \\ &= \frac{\pi_j p_{ji}}{\pi_i} \end{aligned}$$

fortsættes

7.2

17 a fortsat

At kæden i omvendt tid opfylder Markov-betingelsen, ses af

$$P(X_{n-1}=j \mid X_n=i, X_{n+1}=i_1, \dots, X_{n+k}=i_k)$$

$$= \frac{P(X_{n-1}=j, X_n=i, X_{n+1}=i_1, \dots, X_{n+k}=i_k)}{P(X_n=i, X_{n+1}=i_1, \dots, X_{n+k}=i_k)}$$

$$= \frac{P(X_{n+1}=i_1, \dots, X_{n+k}=i_k \mid X_n=i, X_{n-1}=j)\, P(X_n=i, X_{n-1}=j)}{P(X_{n+1}=i_1, \dots, X_{n+k}=i_k \mid X_n=i)\, P(X_n=i)}$$

$$= \frac{P(X_n=i, X_{n-1}=j)}{P(X_n=i)} = P(X_{n-1}=j \mid X_n=i)$$

b Kæden reversibel, når $p_{ij} = q_{ij}$ for alle i,j

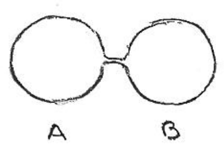
$$p_{ij} = q_{ij} \Leftrightarrow p_{ij} = \frac{\pi_j p_{ji}}{\pi_i} \Leftrightarrow \pi_i p_{ij} = \pi_j p_{ji}$$

c $$\pi_i p_{ij} = \pi_j p_{ji} \Rightarrow \sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji}$$

$$\Leftrightarrow (\pi P)_j = \pi_j 1 \Leftrightarrow \underline{\pi} P = \underline{\pi}$$

dvs. $\underline{\pi}$ er stationær fordeling

20

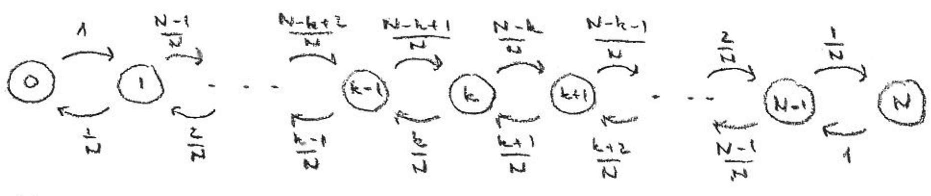


Totalt antal molekyler i to beholdere: N

I hver tidsenhed bevæger der sig tilfældigt ét molekyle fra den ene beholder til den anden.

X_n: Antal molekyler i beholder A

a



b Hvert molekyle har sands. $\frac{1}{2}$ for at være i beholder A, dvs $\pi_k = \binom{N}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{N-k} = \binom{N}{k}\left(\frac{1}{2}\right)^N$

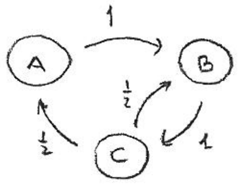
$$\pi_k \pi_{k,k+1} = \pi_{k+1}\pi_{k+1,k} \Leftrightarrow \binom{N}{k}\left(\tfrac{1}{2}\right)^N \frac{N-k}{N} = \binom{N}{k+1}\left(\tfrac{1}{2}\right)^N \frac{k+1}{N}$$

$$\Leftrightarrow \binom{N}{k} = \frac{k+1}{N-k}\binom{N}{k+1} \Leftrightarrow \binom{N}{k} = \binom{N}{k}$$

c $\underline{\pi}$ er ikke grænsefordeling, da kæden er periodisk med perioden 2

SLB

5.7



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

i $R_{AA} = \{3, 5, \dots\}$ $R_{BA} = \{2, 4, 5, \dots\}$ $R_{CA} = \{1, 3, 4, 5, \dots\}$

$R_{AB} = \{1, 3, 4, 5, \dots\}$ $R_{BB} = \{2, 3, 5, \dots\}$ $R_{CB} = \{1, 2, 3, 4, 5, \dots\}$

$R_{AC} = \{2, 4, 5, \dots\}$ $R_{BC} = \{1, 3, 4, 5, \dots\}$ $R_{CC} = \{2, 3, 4, 5, \dots\}$

$\forall (i, j) : \ r_{ij}^{(5)} > 0$

ii $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$ $\forall (i, j) : \ r_{ij}^{(5)} > 0$

8.4

$$P^T - I = \begin{bmatrix} -1 & 0 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r\,(1, 2, 2)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{5}$

$\underline{\pi} = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

$\underline{\mu} = \left(5, \frac{5}{2}, \frac{5}{2}\right)$