7.2

21 $\{X_n\}$ irreducibel, pos. rekurrent m. stat. ford. π

Med $g: S \to \mathbb{R}$ gælder $\frac{1}{n}\sum_{k=1}^{n} g(X_k) \xrightarrow{P} \sum_{j\in S} g(j)\pi_j$

uanset valg af startford.

$$g(i) = P(X_k = i) \Rightarrow \frac{1}{n}\sum_{k=1}^{n} P(X_k = i) \xrightarrow{P} \sum_{j\in S} \delta_{ij}\pi_j = \pi_i$$

7.3

22 Simpelt random walk (for sym. r.w. sæt $p = \frac{1}{2}$)

$P(X_k = 1) = p$, $P(X_k = -1) = 1-p$, X_k'erne uafh.

$$EX_k = 1p + (-1)(1-p) = 2p-1$$

$$\mathrm{Var}\, X_k = EX_k^2 - (EX_k)^2 = 1^2 p + (-1)^2(1-p) - (2p-1)^2 = 1-(2p-1)^2$$

$$S_n = \sum_{k=1}^{n} X_k \qquad S_{n+m} = \sum_{k=1}^{n+m} X_k$$

$$ES_n = n(2p-1) \qquad ES_{n+m} = (n+m)(2p-1)$$

$$\mathrm{Var}\, S_n = n(1-(2p-1)^2) \qquad \mathrm{Var}\, S_{n+m} = (n+m)(1-(2p-1)^2)$$

$$\begin{aligned}\mathrm{Cov}[S_n, S_{n+m}] &= \mathrm{Cov}[S_n, S_n + \sum_{k=n+1}^{n+m} X_k] \\ &= \mathrm{Var}\, S_n + \mathrm{Cov}[\sum_{k=1}^{n} X_k, \sum_{k=n+1}^{n+m} X_k] \\ &= \mathrm{Var}\, S_n + 0 = n(1-(2p-1)^2)\end{aligned}$$

$$\begin{aligned}\rho(S_n, S_{n+m}) &= \frac{n(1-(2p-1)^2)}{\sqrt{n(1-(2p-1)^2)}\sqrt{(n+m)(1-(2p-1)^2)}} \\ &= \frac{n}{\sqrt{n(n+m)}} = \frac{1}{\sqrt{1+\frac{m}{n}}}\end{aligned}$$

$$\rho \to 0 \text{ for } \begin{cases} m \to \infty \\ n \text{ fast} \end{cases}$$

$$\rho \to 1 \text{ for } \begin{cases} n \to \infty \\ m \text{ fast} \end{cases}$$

23 Simpelt random walk, $p \neq \frac{1}{2}$

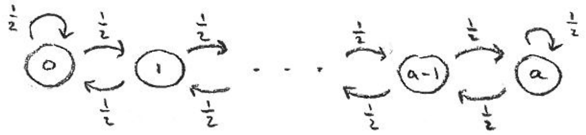
$$ES_n = E\left[\sum_{k=1}^{n} X_k\right] = n(1p + (-1)(1-p)) = n(2p-1)$$

$$\frac{1}{n} S_n \xrightarrow{P} 2p-1 \Rightarrow S_n \xrightarrow{P} \begin{cases} -\infty & \text{for } p < \frac{1}{2} \\ \infty & \text{for } p > \frac{1}{2} \end{cases}$$

7.3

24 Simpelt symmetrisk random walk med reflekterende barrierer i 0 og a.

a



Irreducibel, pos. rekurrent, aperiodisk ($p_{00} > 0$), dvs. ergodisk

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

P er symmetrisk, dvs. den stationære fordeling $\underline{\pi}$ kan nedskrives direkte, jf. opg. 7.2 8.

[Alternativt:

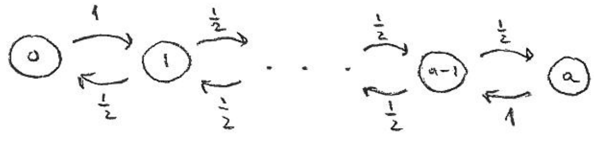
$$P^T - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = \pi_0 (1, 1, \ldots, 1)$, $\sum_i \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{a+1}$]

$\underline{\pi} = \frac{1}{a+1} (1, 1, \ldots, 1)$, også grenseford.

b $E_0(\tau_0) = a+1$

c



Irreducibel, pos. rekurrent, periodisk med perioden 2

fortsettes

7.3

24 c fortsat

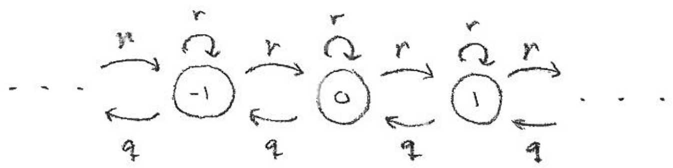
$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = \pi_0\,(1, 2, \ldots, 2, 1)$, $\sum_i \pi_i = 1 \Rightarrow \pi_0 = \dfrac{1}{2+2(a-1)}$

$\underline{\pi} = \frac{1}{2a}(1, 2, \ldots, 2, 1)$, ikke grænsefordeling.

25



i Sæt $u = P_0(\tau_1 < \infty)$

Betingning giver $u = 1p + ru + qu^2$, $r = 1-(p+q)$

$$\Rightarrow u^2 - \left(\frac{p}{q}+1\right)u + \frac{p}{q} = 0 \Rightarrow u = \begin{cases} \frac{p}{q} \\ 1 \end{cases}$$

$$P_0(\tau_1 < \infty) = \begin{cases} \frac{p}{q} & \text{for } p < q \\ 1 & \text{for } p \geq q \end{cases}$$

ii Sæt $\mu = E_0[\tau_1]$

Betingning giver $\mu = 1p + (1+\mu)r + (1+2\mu)q$

$$\Rightarrow (p-q)\mu = 1$$

$$E_0[\tau_1] = \begin{cases} \frac{1}{p-q} & \text{for } p > q \\ \infty & \text{for } p \leq q \end{cases}$$

7.3

26 Simpelt random walk m. $p > \frac{1}{2}$, start i tilst. 0

τ_r : tid til første besøg i tilst. $r \geq 1$

$E_0[\tau_r] = r\, E_0[\tau_1] = r\mu$

Betingning : $\mu = 1\,p + (1+2\mu)\,q \Rightarrow \mu = \frac{1}{p-q}$

$E_0[\tau_r] = \frac{r}{p-q}$

27 Simpelt random walk m. $p \neq \frac{1}{2}$, start i 0

$\tau_0 = \min\{n \geq 1 : S_n = 0\}$, tid til første besøg i 0

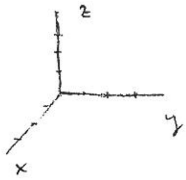
$$P_0(\tau_0 < \infty) = P_1(\tau_0 < \infty)\,p + P_{-1}(\tau_0 < \infty)\,(1-p)$$

$$= \begin{cases} \frac{1-p}{1-(1-p)}\,p + 1\,(1-p) & \text{for } p > \frac{1}{2} \\ 1\,p + \frac{p}{1-p}\,(1-p) & \text{for } p < \frac{1}{2} \end{cases}$$

$$= \begin{cases} 2(1-p) & \text{for } p > \frac{1}{2} \\ 2p & \text{for } p < \frac{1}{2} \end{cases}$$

$$= 2 \min\{1-p, p\}$$

29



Antal skridt langs de tre akser : n

$\underline{S}_n = (S_i^{(x)}, S_j^{(y)}, S_k^{(z)})$, $i+j+k=n$

$\underline{S}_n \sim m(n; \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$

Ved start i $\underline{0}$ må der foretages lige mange skridt i pos. retning og i neg. retning på hver af akserne for at komme tilbage til $\underline{0}$

$$P_{\underline{0}}(\underline{S}_{2n-1} = \underline{0}) = 0$$

$$P_{\underline{0}}(\underline{S}_{2n} = \underline{0}) = \sum_{i+j+k=n} \binom{2n}{i,j,k,i,j,k} \left(\frac{1}{6}\right)^{2n}$$

$$= \sum_{i+j+k=n} \frac{(2n)!}{(i!\,j!\,k!)^2} \left(\frac{1}{6}\right)^{2n} = \sum_{i+j+k=n} \frac{(2n)!}{12^n\, i!\,j!\,k!\,n!} \binom{n}{i,j,k} \left(\frac{1}{3}\right)^n$$

$$\leq \frac{(2n)!}{12^n \left[\left(\frac{n}{3}\right)!\right]^3 n!} \cdot 1 \approx \frac{(2n)^{2n}\sqrt{2n}\, e^{-2n}\sqrt{2\pi}}{12^n \left[\left(\frac{n}{3}\right)^{\frac{n}{3}} \sqrt{\frac{n}{3}}\, e^{-\frac{n}{3}} \sqrt{2\pi}\right]^3 n^n \sqrt{n}\, e^{-n} \sqrt{2\pi}}$$

$= \frac{1}{2}\left(\frac{3}{\pi}\right)^{\frac{3}{2}} n^{-\frac{3}{2}} < \infty$ for alle $n \Rightarrow \underline{0}$ er transient

$\Rightarrow \{S_n\}$ er transient

7.3

31 $P(X=k) = \frac{1}{3}\left(\frac{2}{3}\right)^k, \quad k=0,1,\dots, \quad EX = \mu = 3-1 = 2$

a $E[Z_n] = \mu^n = 2^n$

b $G(s) = \sum_{k=0}^{\infty} s^k \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{2s}{3}\right)^k = \frac{1}{3}\frac{1}{1-\frac{2s}{3}} = \frac{1}{3-2s}$

$G_2(s) = (G\circ G)(s) = \dfrac{1}{3-2\frac{1}{3-2s}} = \dfrac{3-2s}{7-6s}$

$P(Z_2=0) = G_2(0) = \frac{3}{7}$

c $G(s) = s \iff \frac{1}{3-2s} = s \iff 2s^2-3s+1 = 0 \iff s = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$

$q = \frac{1}{2}$

32 Afkom X m. ford. $p_0, p_1, \dots$, sands. fr. br. fkt. $G(s)$, $EX = \mu$

Immigration Y m. ford. $q_0, q_1, \dots$, sands. fr. br. fkt. $H(s)$ $EY = \nu$

$Z_0 = Y_0, \; Y_0 \neq 0 \; ; \; Z_1 = Y_1 + \sum_{k=1}^{Z_0} X_k \; ; \dots ; \; Z_n = Y_n + \sum_{k=1}^{Z_{n-1}} X_k$

a $G_{Z_1}(s) = H(s)\,(H\circ G)(s)$

$P(Z_1=0) = G_{Z_1}(0) = H(0)\,(H\circ G)(0) = q_0\,H(p_0)$

b $Z_n = Z_n^{(0)} + Z_n^{(1)} + \dots + Z_n^{(n)}$

$G_{Z_n}(s) = \prod_{j=0}^{n}(H\circ G_{n-j})(s) = \prod_{j=0}^{n}(H\circ G_j)(s), \quad G_0(s) = s$

$E[Z_n] = G'_{Z_n}(1) = \sum_{k=0}^{n}(H'\circ G_k)(1)\,G_k'(1)\prod_{j\neq k}(H\circ G_j)(1)$

$= \sum_{k=0}^{n} \nu\mu^k\, 1 = \nu\,\frac{1-\mu^{n+1}}{1-\mu}$

c $X \sim b(1,p), \quad G(s) = 1-p+ps, \quad G^{\circ j}(s) = 1-p^j+p^j s$

$Y \sim p(\lambda), \quad H(s) = e^{\lambda(s-1)}$

$G_{Z_n}(s) = \prod_{j=0}^{n}(H\circ G^{\circ j})(s) = \prod_{j=0}^{n} e^{\lambda(1-p^j+p^j s)}$

$= \prod_{j=0}^{n} e^{\lambda p^j (s-1)} = e^{\lambda\frac{1-p^{n+1}}{1-p}(s-1)}$

$Z_n \sim p\left(\lambda\,\frac{1-p^{n+1}}{1-p}\right)$

SLB

9.3 Markovkæde : $p_{i,0} = \frac{1}{i+2}$, $p_{i,i+1} = \frac{i+1}{i+2}$

i $f_{00}^{(n)} = \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n+1}{n} \frac{1}{n+1} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, $n = 1, 2, \ldots$

ii $f_{00} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots$

$= \lim_{N \to \infty} \left(1 - \frac{1}{N+1} \right) = 1$

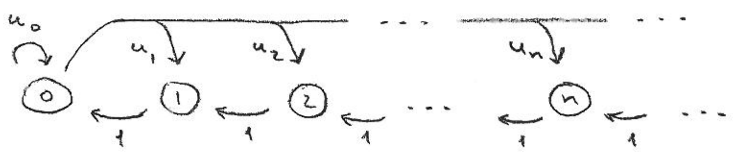
iii $\mu_{00} = \sum_{N=0}^{\infty} P(T_0 > N) = \sum_{N=0}^{\infty} \left(1 - P(T_0 \le N) \right)$

$= \sum_{N=0}^{\infty} \left(1 - \left(1 - \frac{1}{N+1} \right) \right) = \sum_{N=0}^{\infty} \frac{1}{N+1} = \infty$

iv Nulrekurrente

v Ingen stationær fordeling

9.4



$\sum_{j=0}^{\infty} u_j = 1$

i $f_{00} = 1$

ii $\mu_{00} = \sum_{j=0}^{\infty} (1+j) u_j = \sum_{j=0}^{\infty} u_j + \sum_{j=0}^{\infty} j u_j = 1 + \mu$

iii $P = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$

$P^T - I = \begin{bmatrix} u_0 - 1 & 1 & 0 & 0 & \cdots \\ u_1 & -1 & 1 & 0 & \cdots \\ u_2 & 0 & -1 & 1 & \cdots \\ u_3 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sum_1^\infty u_j & -1 & 0 & 0 & \cdots \\ \sum_2^\infty u_j & 0 & -1 & 0 & \cdots \\ \sum_3^\infty u_j & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$

$\underline{\pi} = \pi_0 \left(1, \sum_1^\infty u_j, \sum_2^\infty u_j, \ldots \right)$

$\sum_i \pi_i = 1 \Rightarrow \pi_0 \left(1 + \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} u_j \right) = \pi_0 \left(1 + \sum_{j=1}^{\infty} j u_j \right)$

$= \pi_0 (1 + \mu) = 1 \Rightarrow \pi_0 = \frac{1}{1+\mu}$, $\mu < \infty$

$\underline{\pi} = \frac{1}{1+\mu} \left(1, \sum_{j=1}^{\infty} u_j, \sum_{j=2}^{\infty} u_j, \ldots, \sum_{j=n}^{\infty} u_j, \ldots \right)$, $\mu < \infty$