7.4 *

34 Jf. eks. 7.4.1 s. 442

i Bagudrettet lign.: $P_t' = G P_t$

Fremadrettet lign.: $P_t' = P_t G$; $G = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

ii $P_t' = P_t G \Leftrightarrow P_t'^T = G^T P_t^T$; $G^T = \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix}$

$$\det(G^T - \alpha I) = \begin{vmatrix} -\lambda-\alpha & \mu \\ \lambda & -\mu-\alpha \end{vmatrix} = \dots = \alpha(\alpha + (\lambda+\mu))$$

$$\alpha = 0 : \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix} \sim \begin{bmatrix} \lambda & -\mu \\ 0 & 0 \end{bmatrix}, \quad \bar{v} = r\begin{bmatrix} \mu \\ \lambda \end{bmatrix}, \ r \neq 0$$

$$\alpha = -(\lambda+\mu) : \begin{bmatrix} \mu & \mu \\ \lambda & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{v} = s\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad s \neq 0$$

$$S = \begin{bmatrix} \mu & 1 \\ \lambda & -1 \end{bmatrix}, \quad S^{-1} P_t^T = \begin{bmatrix} c_1 & c_3 \\ c_2 e^{-(\lambda+\mu)t} & c_4 e^{-(\lambda+\mu)t} \end{bmatrix}$$

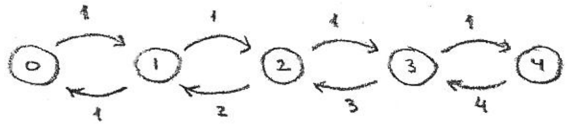
$$P_t^T = \begin{bmatrix} c_1\mu + c_2 e^{-(\lambda+\mu)t} & c_3\mu + c_4 e^{-(\lambda+\mu)t} \\ c_1\lambda - c_2 e^{-(\lambda+\mu)t} & c_3\lambda - c_4 e^{-(\lambda+\mu)t} \end{bmatrix}$$

$$P_0^T = I \Rightarrow P_t^T = \frac{1}{\lambda+\mu}\begin{bmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \mu - \mu e^{-(\lambda+\mu)t} \\ \lambda - \lambda e^{-(\lambda+\mu)t} & \lambda + \mu e^{-(\lambda+\mu)t} \end{bmatrix}$$

$$P_t = \frac{1}{\lambda+\mu}\begin{bmatrix} \mu + \lambda e^{-(\lambda+\mu)t} & \lambda(1 - e^{-(\lambda+\mu)t}) \\ \mu(1 - e^{-(\lambda+\mu)t}) & \lambda + \mu e^{-(\lambda+\mu)t} \end{bmatrix}$$

iii $P_t \to \frac{1}{\lambda+\mu}\begin{bmatrix} \mu & \lambda \\ \mu & \lambda \end{bmatrix}$ for $t \to \infty$

35 a



$X \sim e(1)$

$Y_{(1)} \sim e(n)$

b
$$G = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -4 & 1 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \qquad \left(\Pi = \left(\frac{24}{65}, \frac{24}{65}, \frac{12}{65}, \frac{4}{65}, \frac{1}{65} \right) \right)$$

c $E[Y_{(1)}] = \frac{1}{4}$ $(n = 3+1 = 4)$

d

$P(X < Y_{(1)}) = \frac{1}{1+4} = \frac{1}{5}$

* 33 i'te tilstand i $\{X_t\}$ absorberende $\Rightarrow$ $\gamma_{ij} = 0$ for alle j

$\Rightarrow$ i'te række i G er en nulrække

36 Opholdsintensitet i tilst. i : λ_i (foreløbig)

Antag, at processen ved opholdstidens udløb kan forblive i tilst. i med sands. p_{ii}.

N: antal gange processen successivt befinder sig i tilst. i, $N \sim g(1-p_{ii})$, $N = 1, 2, \dots$

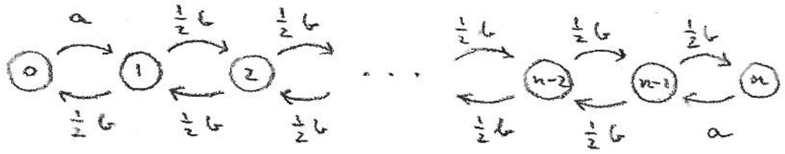
S_N: samlet opholdstid til tilst. i forlades

$S_N \sim e(\lambda_i(1-p_{ii}))$, jf. opg. 3.11 147

Ækvivalent model (den sædvanlige): Sæt $\lambda_i := \lambda_i(1-p_{ii})$, $p_{ii} := 0$ og

$$p_{ij} := \frac{p_{ij}}{1-p_{ii}}, \quad j \neq i$$

37 a.



$$G = \begin{bmatrix} -a & a & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{2}b & -b & \frac{1}{2}b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}b & -b & \frac{1}{2}b & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2}b & -b & \frac{1}{2}b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2}b & -b & \frac{1}{2}b \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a & -a \end{bmatrix}$$

b

$$G^T = \begin{bmatrix} -a & \frac{1}{2}b & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ a & -b & \frac{1}{2}b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}b & -b & \frac{1}{2}b & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2}b & -b & \frac{1}{2}b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2}b & -b & a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}b & -a \end{bmatrix}$$

$$\sim \begin{bmatrix} -2a & b & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -b & b & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & b & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -b & b & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -b & 2a \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

fortsættes

37 b fortsat

$$\underline{\pi} = \pi_0 \left(1, \frac{2a}{b}, \ldots, \frac{2a}{b}, 1\right)$$

$$\sum_i \pi_i = 1 \Rightarrow \pi_0 = \frac{b}{2b + 2a(m-1)}$$

$$\underline{\pi} = \frac{1}{2b + 2a(m-1)} (b, 2a, \ldots, 2a, b)$$

$$b = 2a \Rightarrow \underline{\pi} = \frac{2a}{4a + 2a(m-1)} (1, \ldots, 1)$$

$$= \frac{1}{m+1} (1, \ldots, 1), \text{ dvs. ligeford.}$$

(Sml. : $\underline{\pi}$ i opg. 7.3 24 c er ikke ligeford.)

38 $\{X_n\}$ diskret Markovkæde

$\underline{v}$ vektor med ikke-negative komponenter

Når $\underline{v} P = \underline{v}$, kaldes $\underline{v}$ et invariant mål for $\{X_n\}$

a Antag $0 < \sum_j v_j < \infty$, og betragt $\frac{1}{\sum_j v_j} \underline{v}$

$$\frac{1}{\sum v_j} \underline{v} P = \frac{1}{\sum_j v_j} \underline{v}, \text{ dvs. } \frac{1}{\sum_j v_j} \underline{v} \text{ er stat. ford.}$$

b $\{X_t\}$ Markovkæde i kont. tid m. indlejret $\{X_n\}$

$$\pi_j = \frac{c v_j}{\lambda_j}, \; j \in S, \text{ er løsn. til } \underline{\pi}\, G = 0, \text{ idet}$$

$$\forall j: \sum_i \gamma_{ij} \frac{c v_i}{\lambda_i} = \sum_{i \neq j} \left(-\frac{\gamma_{ij}}{\gamma_{ii}}\right) c v_i - \frac{\gamma_{jj}}{\gamma_{jj}} c v_j$$

$$= \sum_{i \neq j} p_{ij} c v_i - c v_j = c v_j - c v_j$$

$$= 0$$

$$\sum_j \frac{c v_j}{\lambda_j} = 1 \Rightarrow c = \frac{1}{\sum_j \frac{v_j}{\lambda_j}}$$

c $\{X_n\}$ symmetrisk random walk

$\underline{v} = \underline{1}$ er invariant mål for $\{X_n\}$, idet

$$\underline{1} P = \left(\ldots, \tfrac{1}{2} + \tfrac{1}{2}, \ldots\right) = (\ldots, 1, \ldots) = \underline{1}$$

fortsættes

38 fortsat

d Opholdsintensiteter for $\{X_t\}$: $\lambda_0 = 1$

$\lambda_k = k^2$, $k = \pm 1, \pm 2, \dots$

$$\sum_{k=-\infty}^{\infty} \frac{\nu_k}{\lambda_k} = \frac{1}{1} + 2 \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + 2\frac{\pi^2}{6} = \frac{3+\pi^2}{3}$$

Stat. ford. : $\pi_0 = \frac{3}{3+\pi^2}$, $\pi_k = \frac{3}{(3+\pi^2)k^2}$,

$k = \pm 1, \pm 2, \dots$

$\{X_n\}$ er nulrekurrent (jf. sætn. 7.3.1 s. 430)

$\{X_t\}$ er positiv rekurrent

39 Stat. ford. for $\{X_n\}$, jf. opg. 7.2 16, og dermed invariant mål : $\pi_0 = \frac{1}{3}$, $\pi_k = \frac{1}{3} \cdot \frac{1}{2^{k-1}}$, $k = 1, 2, \dots$

Betingelse for at $\{X_t\}$ har stat. ford. : $\sum_{k=0}^{\infty} \frac{\pi_k}{\lambda_k} < \infty$

$$\sum_{k=0}^{\infty} \frac{\pi_k}{\lambda_k} = \frac{1}{3\lambda_0} + \sum_{k=1}^{\infty} \frac{1}{3} \frac{1}{2^{k-1}} \frac{1}{\lambda_k} = \frac{1}{3\lambda_0} + \frac{2}{3} \sum_{k=1}^{\infty} \frac{1}{2^k \lambda_k} < \infty$$

$\lambda_k > (\frac{1}{2})^k$