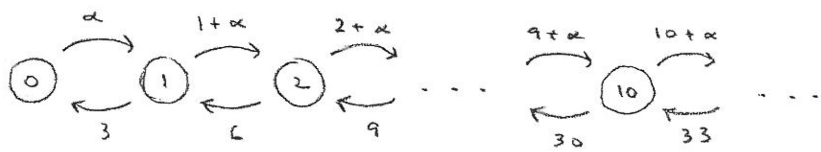
40 a



$$G = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & \cdots \\ 3 & -(4+\alpha) & 1+\alpha & 0 & \cdots \\ 0 & 6 & -(8+\alpha) & 2+\alpha & \cdots \\ 0 & 0 & 9 & -(12+\alpha) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

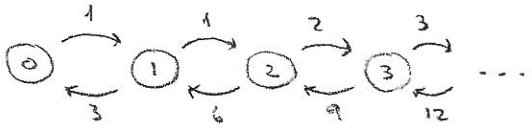
b
$$\frac{10+\alpha}{30+10+\alpha} = \frac{10+\alpha}{40+\alpha}$$

c
$$\alpha = 1$$

$$E_0[\tau_0] = \frac{1}{\pi_0} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot \ldots \cdot n}{3 \cdot (3 \cdot 2) \cdot \ldots \cdot (3n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3}\,\frac{1}{1-\frac{1}{3}} = 1 + \frac{1}{2} = \frac{3}{2}$$

41 $\alpha = 1$



$$G = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots \\ 3 & -4 & 1 & 0 & \cdots \\ 0 & 6 & -8 & 2 & \cdots \\ 0 & 0 & 9 & -12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

fortsættes

7.4

41 fortsat

$$G^T = \begin{bmatrix} -1 & 3 & 0 & 0 & \cdots \\ 1 & -4 & 6 & 0 & \cdots \\ 0 & 1 & -8 & 9 & \cdots \\ 0 & 0 & 2 & -12 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 0 & 0 & \cdots \\ 0 & -1 & 6 & 0 & \cdots \\ 0 & 0 & -2 & 9 & \cdots \\ 0 & 0 & 0 & -3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\pi_1 = \tfrac{1}{3}\pi_0 \quad , \quad \pi_2 = \tfrac{1}{6}\pi_1 = \frac{1}{2\cdot 3^2}\pi_0 \, , \, \ldots \, , \, \pi_n = \frac{1}{n\,3^n}\pi_0 \, , \ldots$$

$$\frac{1}{\pi_0} = 1 + \sum_{n=1}^{\infty}\frac{1}{n\,3^n} = 1 + \sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)^n}{n} = 1 - \ln\left(1-\tfrac{1}{3}\right) = 1 - \ln\tfrac{2}{3}$$

$$\underline{\pi} = \frac{1}{1-\ln\frac{2}{3}}\left(1, \tfrac{1}{3}, \frac{1}{2\cdot 3^2}, \ldots, \frac{1}{n\,3^n}, \ldots\right)$$

42 a

$$\begin{cases} \lambda_0\pi_0 = \mu_1\pi_1 \\ (\lambda_k+\mu_k)\pi_k = \lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1} \quad , \quad k = 1,2,\ldots \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda_0\pi_0 = \mu_1\pi_1 \\ \lambda_{k-1}\pi_{k-1} - (\lambda_k+\mu_k)\pi_k + \mu_{k+1}\pi_{k+1} = 0 \quad , \quad k = 1,2,\ldots \end{cases}$$

$$\Leftrightarrow \sum_{i=0}^{\infty}\pi_i\,\gamma_{ik} = 0 \quad , \quad k = 0,1,\ldots$$

$$\Leftrightarrow \underline{\pi}\,G = \underline{0}$$

b

$$\underline{\pi}\,G = \underline{0} \Leftrightarrow \sum_{i=0}^{\infty}\pi_i\,\gamma_{ij} = 0 \quad , \quad j = 0,1,\ldots$$

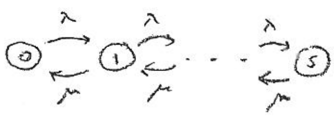
$$\Leftrightarrow \pi_j\,\gamma_{jj} + \sum_{i\neq j}\pi_i\,\gamma_{ij} \quad , \quad j = 0,1,\ldots$$

$$\Leftrightarrow \pi_j\left(-\lambda^{(j)}\right) + \sum_{i\neq j}\pi_i\,\lambda^{(i)}p_{ij} \quad , \quad j = 0,1,\ldots$$

$$\Leftrightarrow \lambda^{(j)}\pi_j = \sum_{i\neq j}\lambda^{(i)}p_{ij}\,\pi_i \quad , \quad j = 0,1,\ldots$$

7.4

43 $M/M/1/5$



a $\mu = \lambda \Rightarrow \rho = 1$

$\pi_5 = \frac{1}{5+1} = \frac{1}{6} = 0{,}1667$, jf. ekv. 7.4.13 s. 458

b $\mu = 2\lambda \Rightarrow \rho = \frac{1}{2}$

$\pi_5 = \frac{(1-\frac{1}{2})(\frac{1}{2})^5}{1-(\frac{1}{2})^6} = \frac{1}{63} = 0{,}0159$, jf. ekv. 7.4.13 s. 458

44 $M/M/k/r$

a $k=1,\ r=1,\ \mu = \lambda \Rightarrow \rho = 1$

$\pi_1 = \frac{1}{1+1} = \frac{1}{2}$

b $k=1,\ r=1,\ \mu = 2\lambda \Rightarrow \rho = \frac{1}{2}$

$\pi_1 = \frac{(1-\frac{1}{2})\frac{1}{2}}{1-(\frac{1}{2})^2} = \frac{1}{3} = 0{,}3333$

c $k=1,\ r=2,\ \mu = \lambda \Rightarrow \rho = 1$

$\pi_2 = \frac{1}{2+1} = \frac{1}{3} = 0{,}3333$

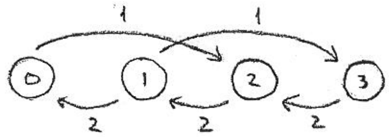
d $k=2,\ r=2,\ \mu = \lambda \Rightarrow \rho = \frac{\lambda}{2\lambda} = \frac{1}{2}$

$\frac{1}{\pi_0} = 1 + \frac{\lambda}{\lambda} + \frac{\lambda \cdot \lambda}{\lambda \cdot 2\lambda} = \frac{5}{2} \Rightarrow \pi_0 = \frac{2}{5}$

$\pi_2 = 2\rho^2 \pi_0 = 2(\frac{1}{2})^2 \frac{2}{5} = \frac{1}{5} = 0{,}2$

7.4

45 a



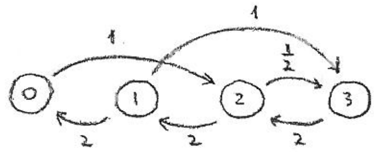
$$G = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$G^T = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & 0 & -2 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r\,(4, 2, 3, 1)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{10}$

$\underline{\pi} = \left(\frac{2}{5}, \frac{1}{5}, \frac{3}{10}, \frac{1}{10}\right)$

b



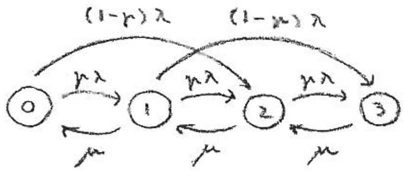
$$G = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 0 & 2 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$G^T = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 0 & -\frac{5}{2} & 2 \\ 0 & 1 & \frac{1}{2} & -2 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 7 & 0 & 0 & -16 \\ 0 & 7 & 0 & -8 \\ 0 & 0 & 7 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r\,(16, 8, 12, 7)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{43}$

$\underline{\pi} = \left(\frac{16}{43}, \frac{8}{43}, \frac{12}{43}, \frac{7}{43}\right)$

46 a



b $\mu = \lambda$, $p = \frac{1}{2}$

$$G = \begin{bmatrix} -\lambda & \frac{1}{2}\lambda & \frac{1}{2}\lambda & 0 \\ \lambda & -2\lambda & \frac{1}{2}\lambda & \frac{1}{2}\lambda \\ 0 & \lambda & -\frac{3}{2}\lambda & \frac{1}{2}\lambda \\ 0 & 0 & \lambda & -\lambda \end{bmatrix} = \lambda \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$G^T \sim \begin{bmatrix} -1 & 1 & 0 & 0 \\ \frac{1}{2} & -2 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 5 & 0 & 0 & -4 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r\,(4, 4, 6, 5)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{19}$

$\underline{\pi} = \left(\frac{4}{19}, \frac{4}{19}, \frac{6}{19}, \frac{5}{19}\right)$

7.4

47 a

$$\begin{tikzcd} 0 \arrow[r, bend left, "3"] & 1 \arrow[l, bend left, "\frac{1}{2}"] \arrow[r, bend left, "{2\cdot 0{,}8+1\cdot 0{,}5 = 2{,}1}"] & 2 \arrow[l, bend left, "\frac{1}{2}"] \end{tikzcd}$$

$$G = \begin{bmatrix} -3 & 3 & 0 \\ 0{,}5 & -2{,}6 & 2{,}1 \\ 0 & 0{,}5 & -0{,}5 \end{bmatrix}$$

$$G^T = \begin{bmatrix} -3 & 0{,}5 & 0 \\ 3 & -2{,}6 & 0{,}5 \\ 0 & 2{,}1 & -0{,}5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 6 & -1 & 0 \\ 0 & 21 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\pi} = r_1\left(\frac{5}{6}, 5, 21\right) = r(5, 30, 126), \quad \sum_i \pi_i = 1 \Rightarrow r = \frac{1}{161}$$

$$\underline{\pi} = \left(\frac{5}{161}, \frac{30}{161}, \frac{126}{161}\right) \approx (0{,}0311;\ 0{,}1863;\ 0{,}7826)$$

b $$\left(\frac{2}{3}\cdot 0{,}2 + \frac{1}{3}\cdot 0{,}5\right)\pi_1 + \pi_2 = 0{,}3\,\frac{30}{161} + \frac{126}{161} = \frac{135}{161} \approx 0{,}8385$$

48 M/M/1 med delvist fravalg af at stå i kø

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\frac{1}{2}\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\frac{1}{3}\lambda"] & \cdots \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\frac{1}{n}\lambda"] & k \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\frac{1}{n+1}\lambda"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$$\sum_{n=1}^{\infty} \frac{\lambda\,\frac{1}{2}\lambda\cdot\frac{1}{3}\lambda\cdot\ldots\cdot\frac{1}{n}\lambda}{\mu^n} = \sum_{n=1}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} = e^{\frac{\lambda}{\mu}} - 1$$

$$\pi_0 = \left(1 + e^{\frac{\lambda}{\mu}} - 1\right)^{-1} = e^{-\frac{\lambda}{\mu}}$$

$$\pi_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}\,\pi_0 = \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^n}{n!}, \quad n = 0, 1, \ldots$$

Poissonsandsynligheder, parameter $\frac{\lambda}{\mu}$

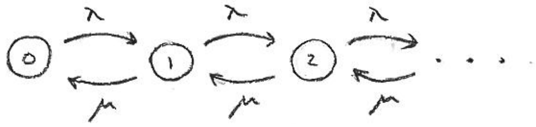
49 M/M/1 med udsivning fra kø

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu+\nu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu+2\nu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "\mu+3\nu"] \end{tikzcd}$$

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & \mu+\nu & -(\lambda+\mu+\nu) & \lambda & \cdots \\ 0 & 0 & \mu+2\nu & -(\lambda+\mu+2\nu) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

7.4

50 M/M/1



W: ventetid i kø

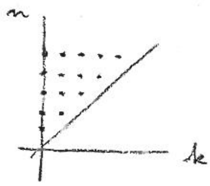
$W_n = \sum_{j=1}^{n} V_j \quad , \quad V_j \sim e(\mu) \quad , \quad j = 1, \ldots, n \qquad$ uafh.

$W_n \sim \Gamma(n, \mu)$

$\pi_n = (1-\rho)\rho^n \quad , \quad \rho = \frac{\lambda}{\mu}$

$P(W=0) = \pi_0 = 1-\rho$

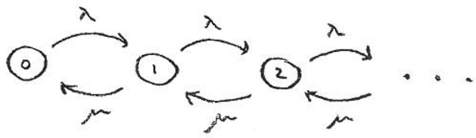
$$
\begin{aligned}
P(W>t) &= \sum_{n=1}^{\infty} P(W_n > t)\, P(X_t = n) \\
&= \sum_{n=1}^{\infty} e^{-\mu t} \sum_{k=0}^{n-1} \frac{(\mu t)^k}{k!} \cdot (1-\rho)\rho^n \qquad \text{(formel nederst s. 137 benyttet)} \\
&= (1-\rho) e^{-\mu t} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(\mu t)^k}{k!} \rho^n \\
&= (1-\rho) e^{-\mu t} \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{(\mu t)^k}{k!} \rho^n \\
&= (1-\rho) e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\rho\mu t)^k}{k!} \sum_{n=k+1}^{\infty} \rho^{n-k} \\
&= (1-\rho) e^{-\mu t} e^{\rho\mu t} \sum_{n=1}^{\infty} \rho^n = (1-\rho) e^{-(1-\rho)\mu t} \frac{\rho}{1-\rho} \\
&= \rho e^{-(1-\rho)\mu t}
\end{aligned}
$$



W's fordeling: $\qquad P(W=0) = 1-\rho$
(hybrid)

$\qquad P(0 < W \le t) = \rho e^{-(1-\rho)\mu t} \quad , \quad 0 < t < \infty$

51 M/M/1



T: ventetid i kø + ekspeditionstid

$T_n = \sum_{j=0}^{n} V_j \quad , \quad V_j \sim e(\mu) \quad , \quad j = 0, 1, \ldots, n \qquad$ uafh.

$T_n \sim \Gamma(n+1; \mu)$

$$
\begin{aligned}
P(T>t) &= \sum_{n=0}^{\infty} P(T_n > t)\, P(X_t = n) \\
&= \sum_{n=0}^{\infty} e^{-\mu t} \sum_{k=0}^{n} \frac{(\mu t)^k}{k!} \cdot (1-\rho)\rho^n , \qquad \text{(formel nederst s. 137 benyttet)}
\end{aligned}
$$

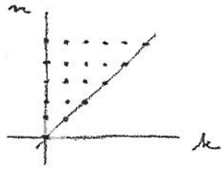
$\rho = \frac{\lambda}{\mu}$

fortsættes

7.4

51 fortsat

$$= (1-\rho)\, e^{-\mu t} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\mu t)^k}{k!} \rho^n$$



$$= (1-\rho)\, e^{-\mu t} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(\mu t)^k}{k!} \rho^n$$

$$= (1-\rho)\, e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\rho \mu t)^k}{k!} \sum_{n=k}^{\infty} \rho^{n-k}$$

$$= (1-\rho)\, e^{-\mu t} e^{\rho \mu t} \sum_{n=0}^{\infty} \rho^n = (1-\rho)\, e^{-(1-\rho)\mu t} \frac{1}{1-\rho}$$

$$= e^{-(1-\rho)\mu t}$$

$$P(T \leq t) = 1 - e^{-(1-\rho)\mu t} \quad , \quad 0 \leq t < \infty$$

$$T \sim e(\,(1-\rho)\mu\,)$$