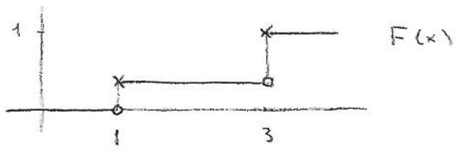
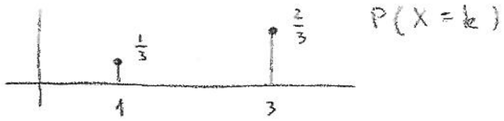
2.2

1





a $F(2) = \frac{1}{3}$

b $P(X>1) = 1 - P(X \le 1)$

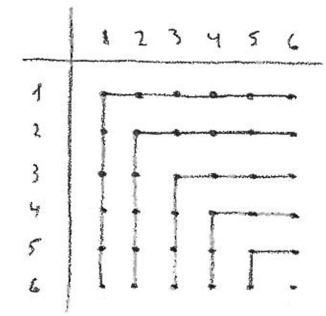
$= 1 - F(1) = 1 - \frac{1}{3} = \frac{2}{3}$

c $P(X=2) = 0$

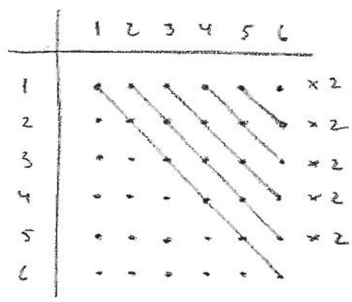
d $P(X=3) = \frac{2}{3}$

2

a



b



k	1	2	3	4	5	6
p_k	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

k	0	1	2	3	4	5
p_k	$\frac{3}{18}$	$\frac{5}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{1}{18}$

$$p_k = \frac{13-2k}{36}, \quad k = 1, \dots, 6$$

$$p_k = \frac{6-k}{18}, \quad k = 1, \dots, 5$$

$$p_0 = \frac{1}{6}$$

2.4

23

a
$$EX = \sum_{k=1}^{6} k \, \frac{13-2k}{36} = \frac{13}{36} \sum_{k=1}^{6} k - \frac{1}{18} \sum_{k=1}^{6} k^2$$

$$= \frac{13}{36} \, \frac{6 \cdot 7}{2} - \frac{1}{18} \, \frac{6 \cdot 7 \cdot 13}{6} = \frac{273}{36} - \frac{91}{18} = \frac{91}{36}$$

b
$$EX = 0 \cdot \frac{1}{6} + \sum_{k=1}^{5} k \, \frac{6-k}{18} = \frac{1}{3} \sum_{k=1}^{5} k - \frac{1}{18} \sum_{k=1}^{5} k^2$$

$$= \frac{1}{3} \, \frac{5 \cdot 6}{2} - \frac{1}{18} \, \frac{5 \cdot 6 \cdot 11}{6} = 5 - \frac{55}{18} = \frac{35}{18}$$

2 2

4 $\quad p_k = \frac{c}{2^k}, \quad k = 0, 1, 2, \ldots$

a $\quad \sum_{k=0}^{\infty} \frac{c}{2^k} = \frac{c}{1-\frac{1}{2}} = 2c = 1 \Rightarrow c = \frac{1}{2} \Rightarrow p_k = \frac{1}{2^{k+1}}$

b $\quad P(X>0) = 1 - P(X=0) = 1 - \frac{1}{2} = \frac{1}{2}$

c $\quad \sum_{j=0}^{\infty} \frac{1}{2^{2j+1}} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{4^j} = \frac{1}{2} \frac{1}{1-\frac{1}{4}} = \frac{2}{3}$

8 $\quad X: \quad p_k = \frac{1}{2^k}, \quad k = 1, 2, 3, \ldots$

$Y = \frac{1}{X}, \quad y = \ldots, \frac{1}{3}, \frac{1}{2}, 1$

$$P(X \le n) = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$$

$$P(Y \le y) = P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right) = 1 - P\left(X < \frac{1}{y}\right)$$

$$= 1 - P\left(X \le \frac{1}{y} - 1\right) = 1 - \left(1 - \left(\frac{1}{2}\right)^{y-1}\right)$$

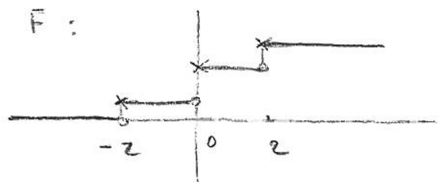
$$= \left(\frac{1}{2}\right)^{y-1} \quad \text{for } y = \frac{1}{n}, \quad n = 1, 2, \ldots$$

$$F_Y(y) = \begin{cases} 0 & \text{for } y \le 0 \\ \left(\frac{1}{2}\right)^{y-1} & \text{for } y = \ldots, \frac{1}{3}, \frac{1}{2} \\ \left(\frac{1}{2}\right)^{\left[\frac{1}{y}\right]} & \text{for } y \in \left]0;1\right[\setminus \{\ldots, \frac{1}{3}, \frac{1}{2}\} \\ 1 & \text{for } y \ge 1 \end{cases}$$

6

k	-2	0	2
p_k	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

F :

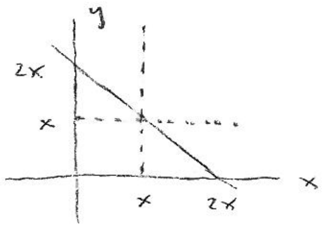


7 a $\quad p_k = \left(\frac{51}{52}\right)^{k-1} \frac{1}{52} = \frac{1}{52} \left(\frac{51}{52}\right)^{k-1}, \quad k = 1, 2, \ldots$

b $\quad p_k = \frac{51}{52} \frac{50}{51} \cdots \frac{52-k+1}{52-k+2} \frac{1}{52-k+1} = \frac{1}{52}, \quad k = 1, \ldots, 52$

2.3

9



$X+Y>2x \supseteq X>x \cap Y>x$

$= S \setminus (X \leq x \cup Y \leq x)$

$X+Y>2x \subseteq X>x \cup Y>x$

$$P(X+Y>2x) \geq 1 - P(X \leq x \cup Y \leq x)$$

$$\geq 1 - \left(P(X \leq x) + P(Y \leq x) \right) \quad *$$

$$= 1 - (F(x) + F(x))$$

$$= 1 - 2F(x)$$

$$P(X+Y>2x) \leq P(X>x \cup Y>x)$$

$$\leq P(X>x) + P(X>x) \quad *$$

$$= 1 - F(x) + 1 - F(x)$$

$$= 2(1-F(x))$$

* jf. opg. 1.3 10 a

12 $\quad f(x) = cx^2, \quad 0 \leq x \leq 1$

a $\quad \int_0^1 cx^2 = c\left[\frac{x^3}{3}\right]_0^1 = c\,\frac{1}{3} = 1 \Rightarrow c = 3 \Rightarrow f(x) = 3x^2,$
$0 \leq x \leq 1$

b $\quad F(x) = \int_0^x 3t^2\,dt = 3\left[\frac{t^3}{3}\right]_0^x = x^3, \quad 0 \leq x \leq 1$

c $\quad Y = \sqrt{X}, \quad y = \sqrt{x} \Rightarrow x = y^2, \quad \frac{dx}{dy} = 2y$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2)$$

$$= (y^2)^3 = y^6, \quad 0 \leq y \leq 1$$

b fortsat

$$P(X > \tfrac{1}{2}) = 1 - F(\tfrac{1}{2}) = 1 - (\tfrac{1}{2})^3 = \tfrac{7}{8}$$

2.3

13 a $f(x) = |x|$, $-1 \le x \le 1$

$$F(x) = \begin{cases} \int_{-1}^{x} -t\,dt = \left[-\frac{t^2}{2}\right]_{-1}^{x} = \frac{1}{2}(1-x^2) & , -1 \le x \le 0 \\ \frac{1}{2} + \int_0^x t\,dt = \frac{1}{2} + \left[\frac{t^2}{2}\right]_0^x = \frac{1}{2}(1+x^2) & , 0 \le x \le 1 \end{cases}$$

b $f(x) = \frac{3}{2}(x^2-1)$, $0 \le x \le 2$

$f(x) < 0$ for $0 \le x < 1$, kan ikke bruges som tæthedsfkt.

c $f(x) = 1$, $-1 \le x \le 0$

$F(x) = x+1$, $-1 \le x \le 0$

d $f(x) = \frac{1}{x^2}$, $1 \le x < \infty$

$F(x) = \int_1^x \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_1^x = 1 - \frac{1}{x}$, $1 \le x < \infty$

15 $f(x) = \frac{1}{x^3}$, $a \le x < \infty$

a $\int_a^\infty \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_a^\infty = \frac{1}{2a^2} = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$

b $F(x) = \int_{\frac{1}{\sqrt{2}}}^{x} \frac{1}{t^3}\,dt = \left[-\frac{1}{2t^2}\right]_{\frac{1}{\sqrt{2}}}^{x} = 1 - \frac{1}{2x^2}$, $\frac{1}{\sqrt{2}} \le x < \infty$

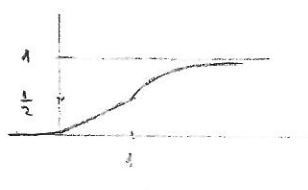
$P(X>3) = 1 - F(3) = \frac{1}{2 \cdot 3^2} = \frac{1}{18}$

c $1 - F(x) = \frac{1}{4} \Leftrightarrow \frac{1}{2x^2} = \frac{1}{4} \Leftrightarrow x^2 = 2 \Leftrightarrow x = \sqrt{2}$

17 $$f(x) = \begin{cases} \frac{1}{2} , & 0 < x \le 1 \\ \frac{1}{2x^2} & 1 \le x < \infty \end{cases}$$

a $$F(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x \le 1 \\ \frac{1}{2} + \left[-\frac{1}{2t}\right]_1^x = 1 - \frac{1}{2x} & \text{for } 1 \le x < \infty \end{cases}$$

b



fortsætter

2.3

17 fortsat

c $\quad Y = \frac{1}{X} \quad , 0 < y < \infty \quad ; \qquad y = \frac{1}{x} \iff x = \frac{1}{y}$

$$f_Y(y) = f_X\left(\frac{1}{y}\right) \left| \frac{d}{dy}\left(\frac{1}{y}\right) \right|$$

$$= \begin{cases} \frac{1}{2} \left| -\frac{1}{y^2} \right| \\ \frac{1}{2\left(\frac{1}{y}\right)^2} \left| -\frac{1}{y^2} \right| \end{cases} = \begin{cases} \frac{1}{2y^2} & \infty > y \geq 1 \\ \frac{1}{2} & 1 \geq y > 0 \end{cases}$$

$$f_Y = f_X$$

19 $\quad X \sim U[0;1]$

a $\quad Y = \frac{1}{X} \quad , \quad y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \quad , \quad \frac{dx}{dy} = -\frac{1}{y^2}$

$$f_Y(y) = 1 \left| -\frac{1}{y^2} \right| = \frac{1}{y^2} \quad , \quad 1 \leq y < \infty$$

b $\quad Y = \frac{1}{\sqrt{X}} \quad , \quad y = \frac{1}{\sqrt{x}} \iff x = \frac{1}{y^2} \quad , \quad \frac{dx}{dy} = -\frac{2}{y^3}$

$$f_Y(y) = 1 \cdot \left| -\frac{2}{y^3} \right| = \frac{2}{y^3} \quad 1 \leq y < \infty$$

c $\quad Y = \ln X \quad , \quad y = \ln x \Rightarrow x = e^y \quad , \quad \frac{dx}{dy} = e^y$

$$f_Y(y) = 1 \left| e^y \right| = e^y \quad , \quad -\infty < y \leq 0$$

20 $\quad f(x) = e^{-x} \quad , \quad 0 \leq x < \infty$

$$Y = e^{-X} \quad , \quad y = e^{-x} \Rightarrow x = -\ln y \quad , \quad \frac{dx}{dy} = -\frac{1}{y}$$

$$f_Y(y) = e^{-(-\ln y)} \left| -\frac{1}{y} \right| = y \frac{1}{y} = 1 \quad , \quad 0 < y \leq 1$$

$$Y \sim U[0;1]$$

21 $\quad f(x) = e^{-x} \quad , \quad 0 \leq x < \infty$

$$Y = \sqrt{X} \quad , \quad y = \sqrt{x} \Rightarrow x = y^2 \quad , \quad \frac{dx}{dy} = 2y$$

$$f_Y(y) = e^{-y^2} \left| 2y \right| = 2y\, e^{-y^2} \quad , \quad 0 \leq y \leq \infty$$

2.4

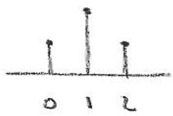
23 Se efter 2.2 2

24 a

$P(X=k)$

$$EX = 0\cdot\frac{1}{8} + 1\cdot\frac{3}{8} + 2\cdot\frac{3}{8} + 3\cdot\frac{1}{8} = \frac{3}{2}$$

b



$P(X=k)$

$$EX = 0\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} = 1$$

25

$$P(X=k) = \frac{\binom{13}{k}\binom{39}{3-k}}{\binom{52}{3}}, \quad k = 0,1,2,3$$

$$EX = \frac{1\cdot 13\cdot\frac{39\cdot 38}{2} + 2\,\frac{13\cdot 12}{2}\cdot 39 + 3\,\frac{13\cdot 12\cdot 11}{3\cdot 2}\cdot 1}{\frac{52\cdot 51\cdot 50}{3\cdot 2}} = \frac{3}{4}$$

$$\left(\text{alt. } EX = 3\cdot\frac{13}{52} = \frac{3}{4}\right)$$