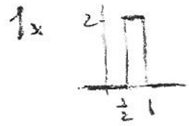
2.4

27 $\quad X \sim G(3, \frac{1}{6})$, $\quad EL = P(X=0) - EX = (\frac{5}{6})^3 - 3 \cdot \frac{1}{6} = \frac{17}{216} \approx 0{,}0787$

32 $\quad X \sim U[\frac{1}{2}; 1]$, $\quad EX = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$



35 $\quad X$, $f(x) = 3x^2$, $0 \le x \le 1$, $\quad Y = \sqrt{X}$

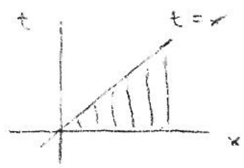
a $\quad EX = \int_0^1 x \, 3x^2 \, dx = \frac{3}{4} \qquad EX^2 = \int_0^1 x^2 \cdot 3x^2 \, dx = \frac{3}{5}$

$\quad \operatorname{Var} X = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$

b $\quad EY = \int_0^1 \sqrt{x} \, 3x^2 \, dx = 3 \cdot \frac{2}{7} = \frac{6}{7} \qquad EY^2 = \int_0^1 x \, 3x^2 \, dx = \frac{3}{4}$

$\quad \operatorname{Var} Y = \frac{3}{4} - \frac{36}{49} = \frac{3}{196}$

36 $\quad X$ ikke-neg. stok. var.



i $\quad EX = \int_0^\infty x f(x) \, dx = \int_0^\infty \int_0^x f(x) \, dt \, dx$

$\quad = \int_0^\infty \int_t^\infty f(x) \, dx \, dt = \int_0^\infty P(X > t) \, dt$

$\quad = \int_0^\infty P(X > x) \, dx$

ii $\quad EX^2 = \int_0^\infty x^2 f(x) \, dx = \int_0^\infty \int_0^x 2t \, f(x) \, dt \, dx$

$\quad = 2 \int_0^\infty \int_t^\infty t \, f(x) \, dx \, dt = 2 \int_0^\infty t \, P(X > t) \, dt$

$\quad = 2 \int_0^\infty x \, P(X > x) \, dx$

37 $\quad X \sim U[0; 1]$, $\quad Y = \frac{1}{X}$, $\quad Z = \frac{1}{\sqrt{X}}$

i $\quad EY = \int_0^1 \frac{1}{x} \cdot 1 \, dx = [\ln x]_0^1 = \infty$, $\quad \operatorname{Var} Y$ ikke def.

ii $\quad EZ = \int_0^1 \frac{1}{\sqrt{x}} \cdot 1 \, dx = [2\sqrt{x}]_0^1 = 2$, $\quad EZ^2 = EY = \infty$, $\quad \operatorname{Var} Z = \infty$

39 $EX = \mu \quad , \quad \operatorname{Var} X = \sigma^2$

a $E[-X] = -EX = -\mu$

$\operatorname{Var}[-X] = (-1)^2 \operatorname{Var} X = \sigma^2$

b $\left.\begin{array}{l} E[aX+b] = a\,EX + b = a\mu + b = 0 \\ \operatorname{Var}[aX+b] = a^2 \operatorname{Var} X = a^2\sigma^2 = 1 \Rightarrow a = \frac{1}{\sigma} \end{array}\right\} \Rightarrow b = -\frac{\mu}{\sigma}$

$aX + b = \frac{X-\mu}{\sigma}$

41 X ikke-neg. stok. var., $EX = \mu$, $\operatorname{Var} X = \sigma^2$, $c, a > 0$

a $P(X \geq c) = \int_c^\infty f(x)\,dx = \frac{1}{c}\int_c^\infty c f(x)\,dx$

$\leq \frac{1}{c}\int_c^\infty x f(x)\,dx \leq \frac{1}{c}\int_0^\infty x f(x)\,dx = \frac{\mu}{c}$

$P(X \geq c) \leq \frac{\mu}{c}$ (Markovs ulighed) talværdi, se nederst

b $P(X \geq \mu + a) = P(X - \mu \geq a) = P(Y \geq a)$, $Y = X - \mu$

$= P(Y + t \geq a + t)$, t reel var.

$\leq P((Y+t)^2 \geq (a+t)^2)$

$\leq \frac{E[(Y+t)^2]}{(a+t)^2} = \frac{E[Y^2 + 2tY + t^2]}{(a+t)^2}$

$= \frac{\sigma^2 + t^2}{(a+t)^2} = g(t)$

Mindsteværdi af $g(t)$:

$g'(t) = \frac{(a+t)^2 2t - (\sigma^2 + t^2) 2(a+t)}{(a+t)^4} = 0$ for

$(a+t)t - (\sigma^2 + t^2) = 0 \Leftrightarrow t = \frac{\sigma^2}{a}$

$g\left(\frac{\sigma^2}{a}\right) = \frac{\sigma^2 + \frac{\sigma^4}{a^2}}{\left(a + \frac{\sigma^2}{a}\right)^2} = \frac{a^2\sigma^2 + \sigma^4}{(a^2+\sigma^2)^2} = \frac{\sigma^2}{a^2+\sigma^2}$

$P(X \geq \mu + a) \leq \frac{\sigma^2}{a^2 + \sigma^2}$ (Cantellis ulighed)

Værdier, jf. eks. 2.4.15 ($P(X \geq 228) \leq 0{,}0137$):

a $P(X \geq 228) \leq \frac{100}{228} = 0{,}4386$

b $P(X \geq 228) = P(X \geq 100 + 128) \leq \frac{15^2}{15^2 + 128^2} = 0{,}0135$

2.5

43 a $I_A = \begin{cases} 1 & \text{når } A \\ 0 & \text{når } A^c \end{cases}$ $\quad I_A^2 = \begin{cases} 1^2 = 1 & \text{når } A \\ 0^2 = 0 & \text{når } A^c \end{cases}$ $\quad I_A^2 = I_A$

b $I_{A^c} = \begin{cases} 1 & \text{når } A^c \\ 0 & \text{når } (A^c)^c = A \end{cases}$ $\quad I_{A^c} = 1 - I_A$

c $I_{A \cap B} = \begin{cases} 1 & \text{når } A \cap B \\ 0 & \text{når } A^c \cup B^c \end{cases}$ $\quad I_A I_B = \begin{cases} 1 \cdot 1 & \text{når } A \cap B \\ \left.\begin{matrix} 0 \cdot 1 \\ 1 \cdot 0 \\ 0 \cdot 0 \end{matrix}\right\} = 0 & \text{når } A^c \cup B^c \end{cases}$

$I_{A \cap B} = I_A I_B$

d $I_{A \cup B} = \begin{cases} 1 & \text{når } A \cup B \\ 0 & \text{når } A^c \cap B^c \end{cases}$ $\quad I_A + I_B - I_{A \cap B} = \begin{cases} \left.\begin{matrix} 1 + 0 - 1 \cdot 0 \\ 0 + 1 - 0 \cdot 1 \\ 1 + 1 - 1 \cdot 1 \end{matrix}\right\} = 1 & \text{når } A \cup B \\ 0 + 0 - 0 \cdot 0 = 0 & \text{når } A^c \cap B^c \end{cases}$

$I_{A \cup B} = I_A + I_B - I_{A \cap B}$

44 a $\binom{10}{0}\left(\frac{5}{6}\right)^{10} = 0,1615$

b $1 - \binom{10}{0}\left(\frac{5}{6}\right)^{10} - \binom{10}{1}\frac{1}{6}\left(\frac{5}{6}\right)^{9} = 0,5155$

c $\binom{10}{0}\left(\frac{5}{6}\right)^{10} + \binom{10}{1}\frac{1}{6}\left(\frac{5}{6}\right)^{9} + \binom{10}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{8} + \binom{10}{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{7} = 0,9303$

46 a $\binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}$ b $\binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1}$ c $\dfrac{\binom{n}{k}\left(\frac{1}{2}\right)^{n}}{1 - \left(\frac{1}{2}\right)^{n}}$

47 a 5 og 5 , $p_5 = 0,2461$

b 4 og 6 , $2p_4 = 2 \cdot 0,2051 = 0,4102$

c $P(X \geq 3) = 0,7734 \quad (X \sim b(7, \frac{1}{2}))$

d $1 - \left(\frac{1}{2}\right)^5 = 0,9688$

e $2\left(\frac{1}{2}\right)^9 = \left(\frac{1}{2}\right)^8 = 0,0039$

49 $X \sim b(n;\ 0,7)$, krav : $P(X > 15) < 0,05$

Tabelarbejde for $n = 17$ giver

$P(X > 14) = P(X \geq 15) = 0,0774 > 0,05$

$P(X > 15) = P(X \geq 16) = 0,0193 < 0,05$

Altså $n = 17$

2.5

53 r'te gunstige udfald indtræffer i k'te forsøg

$$P(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots$$

$X \sim nb(r, p)$

$r = 1: \quad X \sim g(p)$

54 a $X \sim g\left(\frac{1}{5}\right), \quad P(X=5) = \frac{1}{5}\left(\frac{4}{5}\right)^4 = 0{,}4019$

b $X \sim nb\left(5, \frac{1}{5}\right), \quad P(X=10) = \binom{9}{4}\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^5$

$$= 126\left(\frac{2}{5}\right)^{10} = 0{,}0132$$

55 $X \sim p(2), \quad P(X=k) = \frac{e^{-2}\, 2^k}{k!}, \quad k = 0, 1, \dots$

a $P(X \geq 1) = 1 - P(X=0) = 1 - \frac{e^{-2}\, 2^0}{0!} = 1 - e^{-2} \approx 0{,}8647$

b $P(Y=k) = P(X=k \mid X \geq 1) = \frac{P(X=k)}{1-e^{-2}}, \quad k = 1, 2, \dots$

$EY = \frac{EX}{1-e^{-2}} = \frac{2}{1-e^{-2}} \approx 2{,}3130$ ($k=0$ ing. indflydelse)

c $Z \sim g(e^{-2}), \quad EZ = \frac{1}{e^{-2}} = e^2$

$E[Z-1] = e^2 - 1 \approx 6{,}3891$