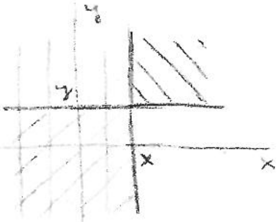
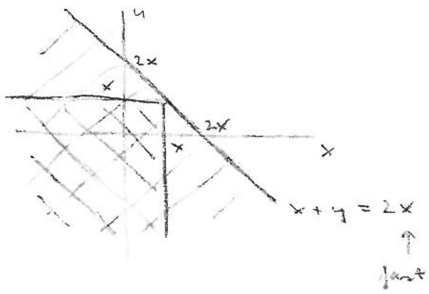
3.2

4 

a $\forall x, y: \quad F(x,y) \le F_X(x)$

b $P(X>x,\ Y>y) \ne 1 - F(x,y)$ i alm.

5 

(X,Y) har ford. fkt. $F(x,y)$

$Z = X+Y \qquad F_Z(z) = G(z)$

$$F(x,x) = P(X \le x,\ Y \le x) \le P(X+Y \le 2x) = G(2x)$$

3.3

12 Træk ét kort, H: en hjerter, A: et es

$$P(A \mid H) = \frac{P(A \cap H)}{P(H)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13} \quad ; \quad P(A) = \frac{4}{52} = \frac{1}{13}$$

A og H er uafh.

a

H \ A: p_{jk}	0	1	p_j
0	$\frac{9}{13}$	$\frac{3}{52}$	$\frac{3}{4}$
1	$\frac{3}{13}$	$\frac{1}{52}$	$\frac{1}{4}$
p_k	$\frac{12}{13}$	$\frac{1}{13}$	1

b

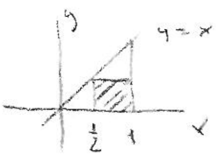
$$P(H \le A) = \frac{36+3+1}{52} = \frac{10}{13}$$

3.4

16 $f(x,y) = c(x-y), \quad 0 \le y \le x \le 1$

a $$\int_0^1 \int_0^x c(x-y)\,dy\,dx = c\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = c\int_0^1 \frac{x^2}{2}\,dx$$

$$= c\left[\frac{x^3}{6}\right]_0^1 = \frac{c}{6} = 1 \Rightarrow c = 6$$

b 

$$P(X > \tfrac{1}{2},\ Y \le \tfrac{1}{2}) = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} 6(x-y)\,dy\,dx$$

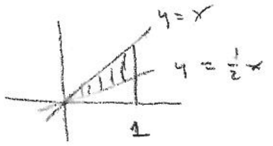
$$= 6\int_{\frac{1}{2}}^1 \left[xy - \frac{y^2}{2}\right]_0^{\frac{1}{2}} dx = 3\int_{\frac{1}{2}}^1 \left(x - \frac{1}{4}\right) dx$$

$$= 3\left[\frac{x^2}{2} - \frac{x}{4}\right]_0^1 = 3\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{4}$$

fortsættes

3.4

16 fortsat

c  $P(X < 2Y) = c \int_0^1 \int_{\frac{x}{2}}^{x} (x-y)\,dy\,dx$

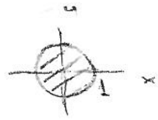
$$= 6 \int_0^1 \left[xy - \frac{y^2}{2} \right]_{\frac{x}{2}}^{x} dx$$

$$= 6 \int_0^1 \left(x^2 - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{8} \right) dx = \frac{3}{4} \int_0^1 x^2\,dx = \frac{3}{4} \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{4}$$

d $$f_X(x) = \int_0^x c(x-y)\,dy = c\left[xy - \frac{y^2}{2} \right]_0^x = 6\left(x^2 - \frac{x^2}{2} \right) = 3x^2, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_y^1 6(x-y)\,dx = 6\left[\frac{x^2}{2} - yx \right]_y^1 = 6\left(\frac{1}{2} - y - \frac{y^2}{2} + y^2 \right)$$

$$= 3 - 6y + 3y^2, \quad 0 \le y \le 1$$

20 $f(x,y) = c\left(1 - (x^2+y^2)\right), \quad x^2+y^2 \le 1$ 

a $$\int_0^{2\pi} \int_0^1 c(1-r^2)\,r\,dr\,d\theta = 2\pi c \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi c}{2} = 1$$

$$\Rightarrow c = \frac{2}{\pi}$$

b $$P(D \le d) = P\left(\sqrt{X^2+Y^2} \le d\right) = \int_0^{2\pi} \int_0^d \frac{2}{\pi}(1-r^2)\,r\,dr\,d\theta$$

$$= \frac{2}{\pi}\, 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^d = 2d^2 - d^4 = d^2(2-d^2)$$

3.5

22 X: antal knækkede æg i første stikprøve, $X \sim h(144, 8, 12)$

Y: antal knækkede æg i anden stikprøve,

$Y \mid X=1 \sim h(132, 7, 12)$

$$P(\text{godkendelse}) = P(X=0) + P(X=1)\,P(Y=0 \mid X=1)$$

$$= \frac{\binom{8}{0}\binom{136}{12}}{\binom{144}{12}} + \frac{\binom{8}{1}\binom{136}{11}}{\binom{144}{12}} \frac{\binom{7}{0}\binom{125}{12}}{\binom{132}{12}} = 0{,}6792$$

(Ved binomialapproksimation: $\left(\frac{17}{18}\right)^{12} + 12\,\frac{1}{18}\left(\frac{17}{18}\right)^{11}\left(\frac{125}{132}\right)^{12} = 0{,}6855$)

25 $X \sim p(\lambda_1)$, $Y \sim p(\lambda_2)$, uafh.

$$P(X+Y=k) = \sum_{j=0}^{k} P(X=j, Y=k-j) = \sum_{j=0}^{k} P(X=j)\,P(Y=k-j)$$

$$= \sum_{j=0}^{k} \frac{e^{-\lambda_1}\lambda_1^j}{j!} \frac{e^{-\lambda_2}\lambda_2^{k-j}}{(k-j)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda_1^j \lambda_2^{k-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^k}{k!}, \quad \text{dvs. } X+Y \sim p(\lambda_1+\lambda_2)$$

fortsættes

3.5

25 fortsat

$$P(X=k \mid X+Y=n) = \frac{P(X=k,\ Y=n-k)}{P(X+Y=n)} = \frac{P(X=k)\ P(Y=n-k)}{P(X+Y=n)}$$

$$= \frac{\dfrac{e^{-\lambda_1}\lambda_1^{k}}{k!}\ \dfrac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^{n}}{n!}} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

dvs. $X \mid X+Y=n \sim b\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

35 X, Y ikke-neg. uafh. kont. stok. var. 

$X < Y$

a $$P(X<Y) = \int_0^\infty P(X<y)\, f_Y(y)\, dy \qquad \text{jf. sætn. } 3.5.2$$

$$= \int_0^\infty F_X(y)\, f_Y(y)\, dy = \int_0^\infty F_X(x)\, f_Y(x)\, dx$$

b $X \sim e(\lambda_1)$, $Y \sim e(\lambda_2)$

$$P(X<Y) = \int_0^\infty \left(1-e^{-\lambda_1 x}\right)\lambda_2 e^{-\lambda_2 x}\, dx$$

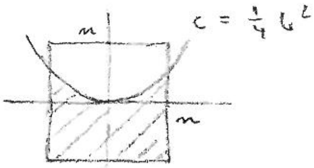
$$= \left[-e^{-\lambda_2 x} + \frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)x}\right]_0^\infty$$

$$= 1 - \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

39 $x^2 + Bx + C$, $B, C \sim U[-n; n]$ uafh., $D = B^2 - 4C$

$P(\text{reelle rødder}) = P(D \geq 0) = 1 - P(D<0)$, $f(b,c) = \frac{1}{4n^2}$

$n \leq 4$:



$$P(D \geq 0) = P\left(C < \tfrac{1}{4}B^2\right)$$

$$= 2\left(\frac{1}{4} + \int_0^n \int_0^{\frac{b^2}{4}} \frac{1}{4n^2}\, dc\, db\right)$$

$$= \frac{1}{2} + \frac{1}{2n^2}\int_0^n \frac{b^2}{4}\, db = \frac{1}{2} + \frac{1}{8n^2}\left[\frac{b^3}{3}\right]_0^n = \frac{1}{2} + \frac{n}{24}$$

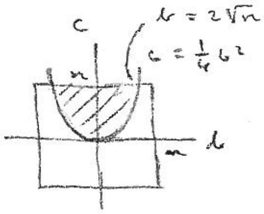
$$P(\text{reelle rødder}) = \frac{1}{2} + \frac{n}{24} \to \frac{1}{2} \text{ for } n \to 0$$

fortsættes

3.5

39 fortsat

$n \geq 4$:



$$P(D<0) = P(C > \tfrac{1}{4}B^2)$$

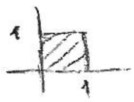
$$= 2\int_0^{2\sqrt{n}}\int_{\frac{b^2}{4}}^{n} \frac{1}{4n^2}\,dc\,db$$

$$= \frac{1}{2n^2}\int_0^{2\sqrt{n}}\left(n - \frac{b^2}{4}\right)db = \frac{1}{2n^2}\left[nb - \frac{b^3}{12}\right]_0^{2\sqrt{n}} = \frac{1}{2n^2}\left(2n\sqrt{n} - \frac{8n\sqrt{n}}{12}\right) = \frac{2}{3\sqrt{n}}$$

$$P(\text{reelle rødder}) = 1 - \frac{2}{3\sqrt{n}} \to 1 \quad \text{for } n \to \infty$$

3.6

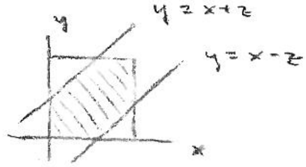
43 $X, Y \sim U[0;1]$ uafh.



$f(x,y) = 1$

a $Z = |X-Y|$, $x - y = z \Leftrightarrow y = x - z$

$x - y = -z \Leftrightarrow y = x + z$

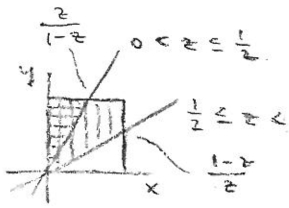


$$F_Z(z) = P(Z \leq z) = P(|X-Y| \leq z) = P(-z \leq X - Y \leq z)$$

$$= 1 - 2 \cdot \tfrac{1}{2}(1-z)^2 = 1 - (1-z)^2, \quad 0 \leq z \leq 1$$

$$f_Z(z) = 2(1-z), \quad 0 \leq z \leq 1$$

b $Z = \frac{X}{X+Y}$, $\frac{x}{x+y} = z \Leftrightarrow y = \frac{1-z}{z}x$



$$F_Z(z) = P(Z \leq z) = P\left(\frac{X}{X+Y} \leq z\right) = P\left(\left(\tfrac{1}{z} - 1\right)X < Y\right)$$

$$= \begin{cases} \frac{1}{2}\cdot 1 \cdot \frac{z}{1-z} = \frac{z}{2(1-z)}, & 0 \leq z \leq \frac{1}{2} \\ 1 - \frac{1}{2}\cdot 1 \cdot \frac{1-z}{z} = \frac{3z-1}{2z}, & \frac{1}{2} \leq z \leq 1 \end{cases}$$

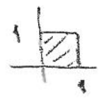
$$f_Z(z) = \begin{cases} \frac{(1-z) - z(-1)}{2(1-z)^2} = \frac{1}{2(1-z)^2}, & 0 \leq z \leq \frac{1}{2} \\ \frac{z \cdot 3 - (3z-1)}{2z^2} = \frac{1}{2z^2}, & \frac{1}{2} \leq z \leq 1 \end{cases}$$

47 $X, Y \sim e(1)$ uafh. , $f(x,y) = e^{-(x+y)}$, $g(X,Y) = e^{-\frac{X+Y}{2}}$

$$E[g(X,Y)] = \int_0^\infty\int_0^\infty e^{-\frac{x+y}{2}}e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty e^{-\frac{3}{2}x}dx \int_0^\infty e^{-\frac{3}{2}y}dy = \frac{2}{3}\cdot\frac{2}{3} = \frac{4}{9}$$

3.6

48 $X, Y \sim U[0,1]$ uafh.  $f(x,y) = 1$

a $E[XY] = \int_0^1 \int_0^1 xy \cdot 1 \, dx\, dy = \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{4}$

b $E\left[\frac{X}{Y}\right] = \int_0^1 \int_0^1 \frac{x}{y} \cdot 1 \, dx\, dy = \left[\frac{x^2}{2}\right]_0^1 \left[\ln y\right]_0^1 = \infty$

c $E[\ln XY] = \int_0^1 \int_0^1 \ln xy \cdot 1 \, dx\, dy = \int_0^1 \int_0^1 (\ln x + \ln y) \cdot dx\, dy$

$= \int_0^1 [x \ln x - x + x \ln y]_0^1 \, dy = \int_0^1 (-1 + \ln y)\, dy$

$= [-y + y \ln y - y]_0^1 = -2$

d $E[|Y - X|] = 2 \int_0^1 \int_0^x (x-y) \cdot 1 \, dy\, dx = 2 \int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$

$= 2 \int_0^1 \frac{x^2}{2}\, dx = \left[\frac{x^3}{3}\right] = \frac{1}{3}$