3.6

50 $X_1, \dots, X_n$ uafh. identisk ford., $EX_i = \mu$, $\operatorname{Var} X_i = \sigma^2$

$\bar{X} = \frac{1}{n} \sum_i X_i$

i $E\bar{X} = \frac{1}{n} \sum_i EX_i = \frac{1}{n} \sum_i \mu = \frac{1}{n} n\mu = \mu$

ii $\operatorname{Var} \bar{X} = \left(\frac{1}{n}\right)^2 \sum_i \operatorname{Var} X_i = \frac{1}{n^2} \sum_i \sigma^2 = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

51 a $E\left[\sum_i X_i^2\right] = \sum_i EX_i^2 = n\left(\operatorname{Var} X_i + (EX_i)^2\right) = n(\sigma^2 + \mu^2)$

b $E\left[\left(\sum_i X_i\right)^2\right] = E\left[\sum_i \sum_j X_i X_j\right] = \sum_i EX_i^2 + 2 \sum_{i<j} EX_i EX_j$

$= n(\sigma^2 + \mu^2) + 2 \frac{n(n-1)}{2} \mu^2 = n(\sigma^2 + n\mu^2)$

52 $X \sim nb(r, p)$, $X = \sum_i X_i$, $X_i \sim g(p)$, $i = 1, \dots, r$, uafh.

i $EX = E\left[\sum_i X_i\right] = \sum_i EX_i = \sum_i \frac{1}{p} = \frac{r}{p}$

ii $\operatorname{Var} X = \operatorname{Var}\left[\sum_i X_i\right] = \sum_i \operatorname{Var} X_i = \sum_i \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$

58 N: antal gifte par, m dødsfald tilfældigt fordelt

X: antal intakte par

I_k: indikatorvar. for ing. dødsfald i k'te par

$$EI_k = \frac{\binom{2N-2}{m}}{\binom{2N}{m}} = \frac{(2N-m)(2N-m+1)}{2N(2N-1)}$$

$$EX = E\left[\sum_i I_k\right] = \sum_i EI_k = N\, EI_k = \frac{(2N-m)(2N-m+1)}{2(2N-1)}$$

61 X, Y ikke-neg. stok. var. m. simultan täth. $f(x,y)$

$$Z = \frac{Y}{X} \qquad \left.\begin{array}{l} x = x \\ z = \frac{y}{x} \end{array}\right\} \iff \left\{\begin{array}{l} x = x \\ y = zx \end{array}\right. \qquad J(x,z) = \begin{vmatrix} 1 & 0 \\ z & x \end{vmatrix} = x$$

a $f_{(X,Z)}(x,z) = f(x, zx)\,|x| = x\, f(x, zx)$, $x, z \geq 0$

$f_Z(z) = \int_0^\infty x\, f(x, zx)\, dx$, $z \geq 0$

fortsættes

3.6

61 fortsat

b $X, Y \sim e(1)$ uafh., $f(x,y) = e^{-(x+y)}$, $Z = \frac{Y}{X}$

$$f(x,z) = x e^{-(x+xz)} = x e^{-(1+z)x}, \quad x, z \geq 0$$

$$f_Z(z) = \int_0^\infty x e^{-(1+z)x} dx$$

$$= \left[x \left(-\frac{1}{1+z} e^{-(1+z)x} \right) \right]_0^\infty + \int_0^\infty \frac{1}{1+z} e^{-(1+z)x} dx$$

$$= 0 + \left[-\frac{1}{(1+z)^2} e^{-(1+z)x} \right]_0^\infty = \frac{1}{(1+z)^2}, \quad z \geq 0$$

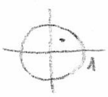
63 $X, Y \sim N(0,1)$ uafh., $f(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$

$U = X - Y$
$V = X + Y$

$$\left.\begin{array}{l} u = x - y \\ v = x + y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(v-u) \end{array}\right. \quad J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$f(u,v) = \frac{1}{2\pi} e^{-\frac{u^2+2uv+v^2+v^2-2uv+u^2}{4\cdot 2}} \left|\frac{1}{2}\right| = \frac{1}{4\pi} e^{-\frac{u^2+v^2}{4}}$$

64



pkt. (r, θ)

$$\left.\begin{array}{l} R \sim U[0;1] \\ \Theta \sim U[0;2\pi] \end{array}\right\} \text{uafh.}, \quad f(r,\theta) = \frac{1}{2\pi}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$J(x,y) = \frac{1}{J(r,\theta)}\Bigg|_{(r,\theta)=\dots} = \frac{1}{r}\Bigg|_{(r,\theta)=\dots} = \frac{1}{\sqrt{x^2+y^2}}$$

$$f(x,y) = \frac{1}{2\pi} \left| \frac{1}{\sqrt{x^2+y^2}} \right| = \frac{1}{2\pi\sqrt{x^2+y^2}}$$

3.7

68 X kont. stok. var. m. tæth. f, $b \in \mathbb{R}$

$$E[X \mid X > b] = \int_{-\infty}^{\infty} x f_{X|X>b}(x) dx$$ jf. def. 3.7.1 s. 189

$$F_{X|X>b}(x) = \frac{F_X(x) - F_X(b)}{P(X > b)}, \quad x \geq b$$

$$f_{X|X>b}(x) = \frac{f_X(x) - 0}{1 - F(b)}, \quad x \geq b$$

$$E[X \mid X > b] = \int_b^\infty x \frac{f_X(x)}{1 - F(b)} dx = \frac{\int_b^\infty x f_X(x) dx}{1 - F(b)}$$

3.7

72 $X \sim U[0;1]$

a $Y|x \sim U[0;x^2]$, $E[Y|x] = \frac{x^2}{2}$

$EY = E[\frac{1}{2}X^2] = \frac{1}{2}EX^2 = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

b $Y|x \sim U[0, \sin\pi x]$, $E[Y|x] = \frac{\sin\pi x}{2}$

$EY = E[\frac{1}{2}\sin\pi X] = \int_0^1 \frac{1}{2}\sin\pi x \cdot 1\,dx = \frac{1}{2}[-\frac{1}{\pi}\cos\pi x]_0^1 = \frac{1}{\pi}$

c $Y|x \sim U[0;\frac{1}{x}]$, $E[Y|x] = \frac{1}{2x}$

$EY = E[\frac{1}{2X}] = \int_0^1 \frac{1}{2x}\cdot 1\,dx = \frac{1}{2}[\ln x]_0^1 = \infty$

d $Y|x \sim e(\frac{1}{x})$, $E[Y|x] = \frac{1}{\frac{1}{x}} = x$

$EY = \int_0^1 x\cdot 1\,dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2}$

73 $X \sim e(1)$

a $Y|x \sim U[0;x]$, $E[Y|x] = \frac{x}{2}$

$EY = \int_0^\infty \frac{x}{2}e^{-x}dx = \frac{1}{2}EX = \frac{1}{2}\cdot 1 = \frac{1}{2}$

b $Y|x \sim U[x;x+2]$, $E[Y|x] = x+1$

$EY = \int_0^\infty (x+1)e^{-x}dx = EX + [-e^{-x}]_0^\infty = 1+1 = 2$

c $Y|x \sim e(x)$, $E[Y|x] = \frac{1}{x}$

$EY = \int_0^\infty \frac{1}{x}e^{-x}dx \geq \int_0^1 \frac{1}{x}e^{-x}dx$

$\geq \int_0^1 \frac{1}{x}\frac{1}{e}dx = \frac{1}{e}[-\frac{1}{x^2}]_0^1 = \infty$

dus. $EY = \infty$

3.7

76 $\quad \mathrm{Var}\, Y = \mathrm{Var}[E[Y|X]] + E[\mathrm{Var}[Y|X]]$

a $\quad X, Y$ uafh.

$$\mathrm{Var}[E[Y|X]] = \mathrm{Var}[EY] = 0$$

b $\quad Y = g(X)$

$$E[\mathrm{Var}[Y|X]] = E[E[Y^2|X] - (E[Y|X])^2]$$
$$= E[(g(X))^2 - (g(X))^2] = 0$$

80 $\quad$ 2 møntkast simultant

Y: antal dobbeltkast indtil HH forekommer i et dobbeltkast

$Y \sim g(\frac{1}{4})$, $\quad EY = \frac{1}{\frac{1}{4}} = 4$

4 dobbeltkast ~ 8 enkeltkast $\quad EX \le 2EY$

$\mu = EX$ i eks. 3.7.8 s. 199 $\quad \Rightarrow \mu \le 8$

84 $\quad$ a

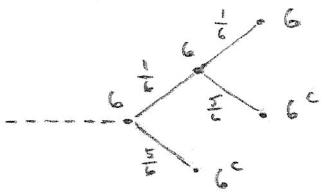
$\dashrightarrow 3 : \; Y \sim g(\frac{1}{6})$

$\dashrightarrow \left\{ \begin{matrix} 6 \\ 6^c \end{matrix} \right. : \; Z \sim g(\frac{5}{6})$

X: antal kast m. terning til sekvensen 3-6

$$EX = EY + (EZ + EX)\tfrac{4}{5} + EZ\,\tfrac{1}{5}$$
$$EX = 6 + (\tfrac{6}{5} + EX)\tfrac{4}{5} + \tfrac{6}{5}\cdot\tfrac{1}{5} \Rightarrow EX = 36$$

b



$\dashrightarrow 6 : \; Y \sim g(\frac{1}{6})$

X: antal kast m. terning til sekvensen 6-6-6

$$EX = \underset{\substack{\uparrow \\ 6}}{EY} + (1+EX)\tfrac{5}{6} + \left((2+EX)\tfrac{5}{6} + 2\cdot\tfrac{1}{6}\right)\tfrac{1}{6}$$

$$EX = 258$$