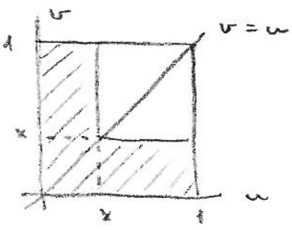
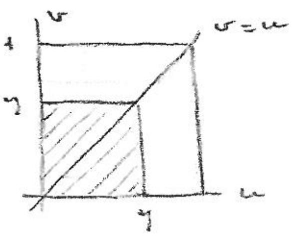
3.8

85 $U, V \sim U[0;1]$ uafh., $X = \min\{U, V\}$
$Y = \text{maks}\{U, V\}$

Fordelingsfunktioner

Bemærk, at $f(u,v) = 1$, $(u,v) \in [0;1] \times [0;1]$





$$F_X(x) = x^2 + 2x(1-x) = 2x - x^2, \quad 0 \le x \le 1$$

$$F_Y(y) = y^2, \quad 0 \le y \le 1$$

eller

$$F_X(x) = P(U \le x) + P(V \le x) - P(U \le x, V \le x)$$
$$= P(U \le x) + P(V \le x) - P(U \le x)P(V \le x)$$
$$= x + x - x \cdot x = 2x - x^2, \quad 0 \le x \le 1$$

$$F_Y(y) = P(U \le y, V \le y) = P(U \le y)P(V \le y)$$
$$= y \cdot y = y^2, \quad 0 \le y \le 1$$

Tæthedsfunktioner

$$f_X(x) = F_X'(x) = 2 - 2x = 2(1-x), \quad 0 \le x \le 1$$

$$f_Y(y) = F_Y'(y) = 2y, \quad 0 \le y \le 1$$

eller

$$f_X(x) = 2 \cdot 1 \cdot (1-x)^{2-1} = 2(1-x), \quad 0 \le x \le 1$$

$f(u) = f(v) = 1$
$F(u) = F(v) = x$

$$f_Y(y) = 2 \cdot 1 \cdot y^{2-1} = 2y, \quad 0 \le y \le 1$$

$f(u) = f(v) = 1$
$F(u) = F(v) = y$

Middelværdier

$$EX = \int_0^1 x \cdot 2(1-x)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

$$EY = \int_0^1 y \cdot 2y\,dy = 2\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3}$$

3.8

85 fortsat

eller

$$EX = \int_0^1 P(X > x)\,dx = \int_0^1 (1 - F_X(x))\,dx = \int_0^1 (1 - (2x - x^2))\,dx$$

$$= \left[x - 2\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 = 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

$$EY = \int_0^1 P(Y > y)\,dy = \int_0^1 (1 - F_Y(y))\,dy = \int_0^1 (1 - y^2)\,dy$$

$$= \left[y - \frac{y^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Kovarians

$$E[XY] = E[UV] = EU\,EV = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$\mathrm{Cov}[X,Y] = E[XY] - EX\,EY = \frac{1}{4} - \frac{1}{3} \cdot \frac{2}{3}$$

$$= \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$

86 I_k : indikatorvar. for match på k'te position

$$E I_k = P(I_k = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$\mathrm{Var}\, I_k = \frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}$$

$$E[I_j I_k] = P(I_j = 1, I_k = 1) = P(I_j = 1 \mid I_k = 1)\, P(I_k = 1)$$

$$= \frac{(n-2)!}{(n-1)!} \frac{(n-1)!}{n!} = \frac{1}{n(n-1)}$$

$$\mathrm{Cov}[I_j, I_k] = \frac{1}{n(n-1)} - \frac{1}{n} \cdot \frac{1}{n} = \frac{n - (n-1)}{n^2(n-1)} = \frac{1}{n^2(n-1)}$$

X : antal match , $\quad EX = \sum_{k=1}^{n} E I_k = n\frac{1}{n} = 1$

$$\mathrm{Var}\, X = \mathrm{Var}\left[\sum_{k=1}^{n} I_k\right] = \sum_{k=1}^{n} \mathrm{Var}\, I_k + 2\sum_{j<k} \mathrm{Cov}[I_j, I_k]$$

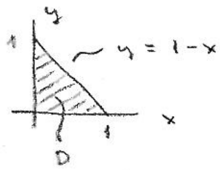
$$= n\frac{n-1}{n^2} + 2\frac{n(n-1)}{2}\frac{1}{n^2(n-1)} = \frac{n-1}{n} + \frac{1}{n} = 1$$

91

$$\rho(X,Y) = \frac{\frac{r(r-N)}{N^2(N-1)}}{\sqrt{\frac{r}{N}\left(1-\frac{r}{N}\right)}\sqrt{\frac{r}{N}\left(1-\frac{r}{N}\right)}} = -\frac{1}{N-1} < 0$$

3.8

92



$(X,Y) \sim U[D]$, $f(x,y) = 2$, $(x,y) \in D$

$EX = \int_0^1 \int_0^{1-x} x \cdot 2 \, dy \, dx = \int_0^1 2x(1-x) \, dx = \left[x^2 - \frac{2x^3}{3} \right]_0^1 = \frac{1}{3}$, $EY = \frac{1}{3}$

$EX^2 = \int_0^1 \int_0^{1-x} x^2 \cdot 2 \, dy \, dx = \int_0^1 2x^2(1-x) \, dx = \left[\frac{2x^3}{3} - \frac{x^4}{2} \right] = \frac{1}{6}$, $EY^2 = \frac{1}{6}$

$E[XY] = \int_0^1 \int_0^{1-x} xy \cdot 2 \, dy \, dx = \int_0^1 x \left[y^2 \right]_0^{1-x} dx = \int_0^1 x(1 - 2x + x^2) \, dx$

$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$

$\mathrm{Var}\, X = \mathrm{Var}\, Y = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$

$\mathrm{Cov}[X,Y] = \frac{1}{12} - \frac{1}{3} \cdot \frac{1}{3} = -\frac{1}{36}$ $\qquad \rho(X,Y) = \dfrac{-\frac{1}{36}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = -\frac{1}{2}$

97

X, Y uafh. , $\mathrm{Var}\, X = \mathrm{Var}\, Y = \sigma^2$, $S = X + Y$, $T = XY$

$ES = EX + EY$, $ET = EX\,EY$

$E[ST] = E[X^2 Y + X Y^2] = EX^2\,EY + EX\,EY^2$

$\mathrm{Cov}[S,T] = EX^2\,EY + EX\,EY^2 - (EX + EY)\,EX\,EY$

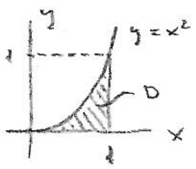
$= EX^2\,EY + EX\,EY^2 - (EX)^2 EY - EX\,(EY)^2$

$= EX\,(EY^2 - (EY)^2) + EY\,(EX^2 - (EX)^2)$

$= EX\,\mathrm{Var}\, Y + EY\,\mathrm{Var}\, X$

$= \sigma^2 (EX + EY) = 0$ for $EX + EY = 0$

98



$(X,Y) \sim U[D]$ $\qquad f(x,y) = k$, $(x,y) \in D$

$1 = \int_0^1 \int_0^{x^2} k \, dy \, dx = \int_0^1 k x^2 \, dx = k \, \frac{1}{3} \Rightarrow k = 3$

$f_X(x) = \int_0^{x^2} 3 \, dy = 3x^2$, $0 \le x \le 1$, $f_{Y|x}(y) = \frac{3}{3x^2} = \frac{1}{x^2}$, $0 \le y \le x^2$

dvs. $Y|x \sim U[0; x^2] \Rightarrow E[Y|x] = \frac{x^2}{2} \Rightarrow E[Y|X] = \frac{X^2}{2}$

$EX = \int_0^1 x \cdot 3x^2 \, dx = \frac{3}{4}$, $EX^2 = \int_0^1 x^2 \cdot 3x^2 \, dx = \frac{3}{5}$

$\mathrm{Var}\, X = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$

$EY = E[E[Y|X]] = \frac{1}{2} EX^2 = \frac{1}{2} \, \frac{3}{5} = \frac{3}{10}$

fortsættes

3.8

98 fortsat

$$E[XY] = \int_0^1 \int_0^{x^2} xy \cdot 3 \, dy \, dx = \int_0^1 3x \frac{x^4}{2} \, dx = \frac{3}{2} \cdot \frac{1}{6} = \frac{1}{4}$$

$$Cov[X,Y] = \frac{1}{4} - \frac{3}{4} \cdot \frac{3}{10} = \frac{1}{40} \quad , \quad \frac{Cov[X,Y]}{Var\, X} = \frac{\frac{1}{40}}{\frac{3}{80}} = \frac{2}{3}$$

$$l(x) = \frac{3}{10} + \frac{2}{3}\left(x - \frac{3}{4}\right) = \frac{3}{10} + \frac{2}{3}x - \frac{1}{2} = \frac{2}{3}x - \frac{1}{5}$$

$$\Rightarrow \quad l(X) = \frac{2}{3}X - \frac{1}{5}$$

101 $X_k \;,\; k = 1, \ldots, n$ uafh. $\quad EX_k = \mu \;,\; Var\, X_k = \sigma^2$

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k \quad , \quad E\bar{X} = \mu \quad , \quad Var\, \bar{X} = \frac{\sigma^2}{n}$$

a

$$Cov[X_j, \bar{X}] = Cov\left[X_j, \frac{1}{n} \sum_k X_k\right]$$

$$= \frac{1}{n}\left(Var\, X_j + \sum_{k \neq j} Cov[X_j, X_k]\right)$$

$$= \frac{1}{n}\left(\sigma^2 + (n-1) \cdot 0\right) = \frac{\sigma^2}{n}$$

$$\rho(X_j, \bar{X}) = \frac{\frac{\sigma^2}{n}}{\sigma \cdot \frac{\sigma}{\sqrt{n}}} = \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} \quad , \quad \rho^2 = \frac{1}{n}$$

b

$$Cov[\bar{X}, X_k - \bar{X}] = Cov[\bar{X}, X_k] - Cov[\bar{X}, \bar{X}]$$

$$= Cov[X_k, \bar{X}] - Var\, \bar{X}$$

$$= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

3.9

103 metalplade $X \times Y$ in , tilstræbt 5×10 in

$$(X,Y) \sim N_2(5 \,;\, 10 \,;\, 0{,}01 \,;\, 0{,}04 \,;\, 0{,}8)$$

$$C = 2(X+Y) \quad , \quad A = XY$$

a

$$EC = 2(EX + EY) = 2(5+10) = 30$$

$$E[C|x] = 2(x + E[Y|x]) = 2\left(x + EY + \rho \frac{\sigma_2}{\sigma_1}(x - EX)\right)$$

$$= 2\left(x + 10 + 0{,}8 \frac{0{,}2}{0{,}1}(x-5)\right) = 5{,}2\,x + 4$$

fortsættes

3.9

103 fortsat

b $\quad EA = E[XY] = Cov[XY] + EXEY = \rho\sigma_1\sigma_2 + \mu_1\mu_2$

$$= 0{,}8 \cdot 0{,}1 \cdot 0{,}2 + 5 \cdot 10 = 50{,}016$$

$$E[A|x] = E[XY|x] = x\,(10 + 1{,}6\,(x-5)) = 1{,}6\,x^2 + 2x$$

c $\quad 29 < C < 31 \iff |C - 30| \leq 1$

$$C \sim N(\,2(5+10)\,;\; 2^2 \cdot 0{,}01 + 2 \cdot 2 \cdot 2 \cdot 0{,}8 \cdot 0{,}1 \cdot 0{,}2 + 2^2 \cdot 0{,}04\,)$$

$$= N(30\,;\; 0{,}328)$$

$$P(|C-30| \leq 1) = 2\,\Phi\left(\frac{1}{\sqrt{0{,}328}}\right) - 1 = 2 \cdot 0{,}95962 - 1$$

$$= 0{,}9192$$

$$P(\text{pladen kasseres}) = 1 - 0{,}9192 = 0{,}0808$$

d $\quad x = 5{,}1 \;;\quad C|x = 2(5{,}1 + Y|x) = 2Y|x + 10{,}2$

$$C|x \sim N(5{,}2 \cdot 5{,}1 + 4\,;\; 2^2 \cdot 0{,}04\,(1 - 0{,}8^2)\,)$$

$$= N(30{,}52\,;\; 0{,}0576) = N(30{,}52\,;\; 0{,}24^2)$$

$$29 < C|x < 31 \iff -1{,}52 < C|x - \mu < 0{,}48$$

$$P(29 < C|x < 31) = \Phi\left(\frac{0{,}48}{0{,}24}\right) - \Phi\left(\frac{-1{,}52}{0{,}24}\right)$$

$$= 0{,}9772 - 0{,}0000 = 0{,}9772$$

$$P(\text{pladen kasseres}) = 1 - 0{,}9772 = 0{,}0228$$

e $\quad C \sim N(30\,;\; 0{,}328\,c)$

$$2\,\Phi\left(\frac{1}{\sqrt{0{,}328\,c}}\right) - 1 > 1 - 0{,}01 = 0{,}99$$

$$\iff \Phi\left(\frac{1}{\sqrt{0{,}328\,c}}\right) > 0{,}995$$

$$\iff \frac{1}{\sqrt{0{,}328\,c}} > 2{,}5757$$

$$\iff c < 0{,}4606$$

3.9

105 $X \sim N(5,1)$, $Y \sim N(7,1)$ ανεξ.

$X - Y \sim N(-2, 2)$, $Y - 2X \sim N(-3, 5)$

α $P(X \geq Y) = P(X - Y \geq 0) = 1 - \Phi\left(\frac{-(-2)}{\sqrt{2}}\right)$

$= 1 - \Phi(\sqrt{2}) = 1 - 0{,}9213 = 0{,}0787$

β $P(Y \geq 2X) = P(Y - 2X \geq 0) = 1 - \Phi\left(\frac{-(-3)}{\sqrt{5}}\right)$

$= 1 - 0{,}9104 = 0{,}0896$

c $\bar{X} \sim N(5, \frac{1}{n})$, $\bar{Y} \sim N(7, \frac{1}{n})$, $\bar{X} - \bar{Y} \sim N(-2, \frac{2}{n})$

$P(\bar{Y} \geq \bar{X}) = P(\bar{X} - \bar{Y} \leq 0) = \Phi\left(\frac{-(-2)}{\sqrt{\frac{2}{n}}}\right) = \Phi(\sqrt{2n})$

$\Phi(\sqrt{2n}) > 0{,}99 \Leftrightarrow \sqrt{2n} > 2{,}3263 \Rightarrow n > 2{,}7 \Rightarrow n \geq 3$

107 $(X,Y) \sim N_2(0,0,1,1,\rho)$, $U = X+Y$, $V = X - Y$

$$\left.\begin{array}{l} u = x+y \\ v = x-y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{array}\right. \qquad J(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 + 2\rho x y + y^2)\right)$$

$$f_{(U,V)}(u,v) = f\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right) \left|-\tfrac{1}{2}\right|$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\tfrac{1}{4}(u^2 + 2uv + v^2) + 2\rho\tfrac{1}{4}(u^2 - v^2) + \tfrac{1}{4}(u^2 - 2uv + v^2)\right)\right)$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{4(1-\rho^2)}\left(u^2 + \rho(u^2 - v^2) + v^2\right)\right)$$

$$= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{4(1+\rho)(1-\rho)}\left((1+\rho)u^2 + (1-\rho)v^2\right)\right)$$

$$= \frac{1}{4\pi\sqrt{(1+\rho)(1-\rho)}} \exp\left(-\frac{u^2}{4(1-\rho)} - \frac{v^2}{4(1+\rho)}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2(1-\rho)}} \exp\left(-\frac{u^2}{4(1-\rho)}\right) \frac{1}{\sqrt{2\pi}\sqrt{2(1+\rho)}} \exp\left(-\frac{v^2}{4(1+\rho)}\right)$$

$\Rightarrow U \sim N(0, 2(1-\rho))$, $V \sim N(0, 2(1+\rho))$ ανεξ.

3.9

108 $X \sim N(1000, 100)$, $Y \sim N(800, 81)$ uafh.

a $Z = \frac{1}{2}(X+Y)$, $Z \sim N(900, \frac{181}{4})$

$$f_Z(z) = \frac{2}{\sqrt{181}} \varphi\left(\frac{2(z-900)}{\sqrt{181}}\right), \quad \varphi = \Phi'$$

b I : indikatorvar. for valg af type 1

$$W = IX + (1-I)Y$$

$$\begin{aligned}
F_W(w) &= P(W \le w) \\
&= P(W \le w \mid I=1)\,P(I=1) + P(W \le w \mid I=0)\,P(I=0) \\
&= P(X \le w)\tfrac{1}{2} + P(Y \le w)\tfrac{1}{2} \\
&= \tfrac{1}{2}F_X(w) + \tfrac{1}{2}F_Y(w) \\
&= \tfrac{1}{2}\Phi\left(\frac{w-1000}{10}\right) + \tfrac{1}{2}\Phi\left(\frac{w-800}{9}\right)
\end{aligned}$$

$$f_W(w) = \frac{1}{20}\varphi\left(\frac{w-1000}{10}\right) + \frac{1}{18}\varphi\left(\frac{w-800}{9}\right)$$

W er <u>ikke</u> normalfordelt

c $EZ = \frac{1}{2}(EX + EY) = 900$

$$EW = \tfrac{1}{2}EX + \tfrac{1}{2}EY = 900$$

$$\operatorname{Var} Z = \tfrac{1}{4}(\operatorname{Var} X + \operatorname{Var} Y) = \frac{181}{4}$$

$$\begin{aligned}
EW^2 &= \int_{-\infty}^{\infty} w^2 f_W(w)\,dw \\
&= \frac{1}{20}\int_{-\infty}^{\infty} w^2 \varphi\left(\frac{w-1000}{10}\right)dw + \frac{1}{18}\int_{-\infty}^{\infty} w^2 \varphi\left(\frac{w-800}{9}\right)dw \\
&= \frac{10}{20}\int_{-\infty}^{\infty}(10u+1000)^2\varphi(u)\,du + \frac{9}{18}\int_{-\infty}^{\infty}(9v+800)^2\varphi(v)\,dv \\
&= \tfrac{1}{2}(100\cdot 1 + 20.000\cdot 0 + 1.000.000\cdot 1) \\
&\quad + \tfrac{1}{2}(81\cdot 1 + 14.400\cdot 0 + 640.000\cdot 1) \\
&= 820.090,5
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var} W &= 820.090,5 - 900^2 = 820.090,5 - 810.000 \\
&= 10.090,5
\end{aligned}$$