Punktestimater

$\tilde{\Theta} = g(y)$ estimat / estimator

Momentmetoden

Y én-dim., $f(\cdot\,;\theta)$ θ k-dim

$$E[Y] = g_1(\theta_1, \dots, \theta_k)$$
$$\vdots$$
$$E[Y^k] = g_k(\theta_1, \dots, \theta_k)$$

$E[Y^j]$ erstattes af tilsvarende stikprøvemoment i pågældende lign.

$$\left.\begin{aligned} g_1(\theta_1, \dots, \theta_k) &= m_1 \\ &\vdots \\ g_k(\theta_1, \dots, \theta_k) &= m_k \end{aligned}\right\}$$

heraf bestemmes $\tilde{\theta}_j$, $j = 1, \dots, k$

eks. $f(y;\theta) = e^{-(y-\theta)}$, $\theta < y < \infty$, θ én-dim

$$E[Y] = \int_\theta^\infty y\, e^{-(y-\theta)}\,dy$$
$$= [y(-e^{-(y-\theta)})]_\theta^\infty + \int_\theta^\infty e^{-(y-\theta)}\,dy$$
$$= 0 + \theta + [-e^{-(y-\theta)}]_\theta^\infty = \theta + 1$$
$$\theta + 1 = \bar{y} \Rightarrow \hat{\theta} = \bar{y} - 1$$

□

ulemper

- stikprøvens elementer skal have samme fordeling
- k stor kan give ustabile estimatorer

Maksimum likelihood estimater

$\hat{\theta}$ vælges, så $L(\hat{\theta}) = \max_{\theta \in \Theta} L(\theta)$

'Størst vægt på det, som er observeret'

Om metoden

- ikke nødvendigvis parametrisk model
- MLE eksisterer ikke altid
- MLE ikke nødvendigvis entydig
- maksimering skal foregå over Θ som specificeret af modellen, ikke nødvendigvis størst mulige område
- MLE kan normalt ikke bestemmes eksplicit, men numeriske metoder må anvendes
- når $t(y)$ er sufficient for θ, så er $\hat{\theta}$ en fkt. af $t(y)$, idet

$$L(\theta; y) = c(y)\, g(t(y); \theta)$$

Men ofte

- MLE eksisterer og er entydig!

eks. $y = (y_1, \dots, y_n)$, $Y_i \sim N(\theta, 1)$ uafh.

$$L(\theta; y) = \exp(-\tfrac{1}{2} \underbrace{\textstyle\sum_i (y_i - \theta)^2}_{h(\theta; y)})$$

maksimering af L ~ minimering af h

$$h(\theta; y) = \sum_i (y_i - \theta)^2 = \sum_i y_i^2 - 2\theta \sum_i y_i + n\theta^2$$

a $-\infty < \theta < \infty$

h har min. for $\hat{\theta} = \frac{1}{n} \sum_i y_i = \bar{y}$

b $0 \leq \theta < \infty$

$\bar{y} > 0$: som under a

$\bar{y} < 0$: h har min. for $\hat{\theta} = 0$

c $0 < \theta < \infty$

$\bar{y} > 0$: som under a og b

$\bar{y} < 0$: MLE eks. ikke, men ... □

eks. $y = (y_1, \dots, y_n)$, $f(y_i;\theta) = \frac{1}{2} e^{-|y_i - \theta|}$, uafh. obs.

$$L(\theta;y) = c \prod_i e^{-|y_i - \theta|} = c \exp\Big(- \underbrace{\sum_i |y_{(i)} - \theta|}_{h(\theta;y)}\Big)$$

maks L ~ min h

h er konveks, da sum af konvekse fkt.'er

a n ulige, $n = 2k-1$

$$h(y_{(k)}) = \sum_{j=1}^{k-1} (y_{(k)} - y_{(j)}) + \sum_{j=k+1}^{2k-1} (y_{(j)} - y_{(k)})$$

vælg $\varepsilon > 0$, så $y_{(k)} + \varepsilon < y_{(k+1)}$

$$h(y_{(k)} + \varepsilon) = h(y_{(k)}) + k\varepsilon - (k-1)\varepsilon$$
$$= h(y_{(k)}) + \varepsilon > h(y_{(k)})$$

vælg $\varepsilon > 0$, så $y_{(k)} - \varepsilon > y_{(k-1)}$

$$h(y_k - \varepsilon) = h(y_{(k)}) - (k-1)\varepsilon + k\varepsilon$$
$$= h(y_{(k)}) + \varepsilon > h(y_{(k)})$$

dvs. $\hat{\theta} = y_{(k)}$, da h er konveks

b n lige, $n = 2k$

betragt θ_1 og $\theta_2 = \theta_1 + \varepsilon$, $\varepsilon > 0$, så

$$y_{(k)} < \theta_1 < y_{(k+1)}, \quad y_{(k)} < \theta_2 < y_{(k+1)}$$

$$h(\theta_2) = h(\theta_1) + k\varepsilon - k\varepsilon = h(\theta_1)$$

betragt θ_1 og $\theta_2 = \theta_1 + \varepsilon$, $\varepsilon > 0$, så

$$y_{(k)} < \theta_1 < y_{(k+1)}, \quad y_{(k+1)} < \theta_2 < y_{(k+2)}$$

$$h(\theta_2) = \sum_{j=1}^{k+1}(\theta_1+\varepsilon - y_{(j)}) + \sum_{j=k+2}^{2k}(y_{(j)} - (\theta_1+\varepsilon))$$

$$= \sum_{j=1}^{k}(\theta_1+\varepsilon-y_{(j)}) + \theta_1+\varepsilon - y_{(k+1)}$$

$$+ \sum_{j=k+1}^{2k}(y_{(j)}-(\theta_1+\varepsilon)) - (y_{(k+1)} - (\theta_1+\varepsilon))$$

$$= h(\theta_1) + \underbrace{2\varepsilon - 2(y_{(k+1)}-\theta_1)}_{>0} > h(\theta_1)$$

vælg θ_1 og $\theta_2 = \theta_1 - \varepsilon$, $\varepsilon > 0$, så

$y_{(k)} < \theta_1 < y_{(k+1)}$, $y_{(k-1)} < \theta_2 < y_{(k)}$

$$h(\theta_2) = \sum_{j=1}^{k-1}(\theta_1-\varepsilon-y_{(j)}) + \sum_{j=k}^{2k}(y_{(j)}-(\theta_1-\varepsilon))$$

$$= \sum_{j=1}^{k}(\theta_1-\varepsilon-y_{(j)}) - (\theta_1-\varepsilon-y_{(k)})$$

$$+ \sum_{j=k+1}^{2k}(y_{(j)}-(\theta_1-\varepsilon)) + y_{(k)} - (\theta_1-\varepsilon)$$

$$= h(\theta_1) + \underbrace{2\varepsilon - 2(\theta_1 - y_{(k)})}_{>0} > h(\theta_1)$$

dvs alle θ, $y_{(k)} \le \theta \le y_{(k+1)}$, er MLE

sammenfatning: $\hat{\theta}$ er stikprøvemedianen □

eks. $(y_1,\dots,y_n)$, $Y_i \sim U(0,\theta)$ uafh.

$$L(\theta;y) = \frac{1}{\theta^n} I_{(y_{(n)},\infty)}(\theta) \quad \text{jf. tidl. eks}$$

aft., dvs. $\hat{\theta} = y_{(n)}$ □

eks. $(y_1,\dots,y_n)$, $Y_i \sim U(\theta, 2\theta)$

$$L(\theta) = \frac{1}{\theta^n} I_{\left(\frac{y_{(n)}}{2},\, y_{(1)}\right)}(\theta) \quad \text{jf. tidl. eks}$$

aft, dvs. $\hat{\theta} = \frac{y_{(n)}}{2}$

bemærk, at $\hat{\theta}$ ikke er suff., men fkt. af suff. ... □

eks. N, antal fisk i sø, skal estimeres.

Fang først M fisk og mærk dem,

fang senere n fisk og noter antal mærkede fisk, m. Der gælder

$$P(Y=m) = \frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}} \ *, \quad \text{maks}\{0, n-N+M\} \le m \le \min\{n, M\}$$

* nødv. bet.: $m \le M$ og $n-m \le N-M$

$$= \frac{M!\,(N-M)!\,n!\,(N-n)!}{m!\,(M-m)!\,(n-m)!\,(N-M-n+m)!\,N!}$$

$$= L(N), \quad N \ge M+n-m$$

L er voks. for $\frac{L(N)}{L(N-1)} > 1 \Rightarrow N < \frac{nM}{m}$

L er aft. for $\frac{L(N)}{L(N-1)} < 1 \Rightarrow N > \frac{nM}{m}$

MLE er derfor $\left[\frac{nM}{m}\right]$ eller $\left[\frac{nM}{m}\right]+1$

Binomialapproksimation, når $N \gg n$:

$$L(p;m) = \binom{n}{m} p^m (1-p)^{n-m}, \quad p = \frac{M}{N}$$

$$l(p;m) = c + m\ln p + (n-m)\ln(1-p)$$

$$\frac{dl}{dp} = \frac{m}{p} - \frac{n-m}{1-p} = 0 \text{ for } \hat{p} = \frac{m}{n}$$

heraf $\frac{M}{\tilde{N}} = \frac{m}{n} \Rightarrow \tilde{N} = \frac{nM}{m}$ $\square$

Ækvivarians

$\psi: \Theta \to \Psi$ bijektion, $\psi = \psi(\theta)$

invers afb.: $\theta = \theta(\psi)$

$L_\Psi(\psi) = L(\theta(\psi))$

L_ψ har maks. for $\hat\psi = \psi(\hat\theta)$, idet

$$L_\psi(\psi(\hat\theta)) = L(\hat\theta) > L(\theta)\big|_{\theta \neq \hat\theta}$$

eks $L(\theta; y) = \exp\left(-\frac{1}{2}\sum_i (y_i - \theta)^2\right)$ $\quad (Y_i \sim N(\theta, 1)$ uafh.)

$$\psi = P(Y_i < 0) = P\left(\frac{Y_i - \theta}{1} < \frac{0-\theta}{1}\right) = \Phi(-\theta)$$

hvoraf $\hat\psi = \Phi(-\hat\theta)$ □

eks. $Y = b(n, \theta)$

$\psi = \frac{1}{n}\operatorname{Var} Y = \theta(1-\theta)$ ikke entydig

$\hat\psi = \hat\theta(1-\hat\theta)$ pr. def. — eller benyt

induceret likelihood $L_\psi(\psi_0) = \max_{\{\theta \mid \psi(\theta) = \psi_0\}} L(\theta)$ □

Likelihood ligninger

ln er voks. fkt $\Rightarrow$

$\hat\theta$ kan findes ved maksimering af

$\ell(\theta; y) = \ln L(\theta; y)$

likelihood ligninger : $\frac{d}{d\theta}\ell(\theta; y) = 0$

betingelser ikke tilstrækkelige

eks. $(y_1, \dots, y_n)$, $Y_i \sim N(\mu, \sigma^2)$ uafh.

$$\ell(\theta; y) = -\frac{n}{2}\ln\sigma^2 - \sum_i \frac{(y_i - \mu)^2}{2\sigma^2} \qquad (\theta = (\mu, \sigma^2))$$

$$= -\frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\left(\sum_i y_i^2 - 2\mu\sum_i y_i + n\mu^2\right)$$

$$\frac{\partial\ell}{\partial\mu} = \frac{1}{\sigma^2}\left(\sum_i y_i - n\mu\right) \qquad (1)$$

$$\frac{\partial\ell}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\left(\sum y_i^2 - 2\mu\sum_i y_i + n\mu^2\right) \qquad (2)$$

(1) $\hat{\mu} = \bar{y} = m_1$

(2) $\hat{\sigma}^2 = \frac{1}{n}\sum_i y_i^2 - \bar{y}^2 = m_2 - m_1^2$

$$\frac{d^2}{d\theta\, d\theta^T} l(\theta)\Big|_{\theta=\hat\theta} = \begin{bmatrix} -\frac{n}{\hat\sigma^2} & 0 \\ 0 & -\frac{n}{2\hat\sigma^4} \end{bmatrix}$$ negativ definit

dvs. $l(\theta)$ er konkav i $\hat\theta$

$l(\theta) \to -\infty$ for $\theta \to$ randen

altså $\hat\theta = \left(\bar y, \frac{1}{n}\sum y_i^2 - \bar y^2\right)$

□

Eks autoregressiv proces af 1. orden, tidl.:

$$2l(\rho,\sigma^2) = c + \ln(1-\rho^2) - n\ln\sigma^2 - \frac{1}{\sigma^2}\left(d_{00} - 2\rho d_{01} + \rho^2 d_{11}\right)$$

$$2\frac{\partial l}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4}\left(d_{00} - 2\rho d_{01} + \rho^2 d_{11}\right) = 0 \qquad (1)$$

$$2\frac{\partial l}{\partial \rho} = \frac{-2\rho}{1-\rho^2} - \frac{1}{\sigma^2}\left(-2d_{01} + 2\rho d_{11}\right) = 0 \qquad (2)$$

(1) $\hat\sigma^2 = \frac{1}{n}\left(d_{00} - 2\hat\rho d_{01} + \hat\rho^2 d_{11}\right)$

(2) $\frac{\hat\rho}{1-\hat\rho^2} - \frac{n}{d_{00} - 2\hat\rho d_{01} + \hat\rho^2 d_{11}}\left(d_{01} + \hat\rho^2 d_{11}\right) = 0$

$(n-1)d_{11}\hat\rho^3 - (n-2)d_{01}\hat\rho^2 - (d_{00} + n d_{11})\hat\rho + n d_{01} = 0$

venstre side : $c(\hat\rho)$

bemærk : $c(1) = (n-1)d_{11} - (n-2)d_{01} - d_{00} - nd_{11} + nd_{01}$

$= -d_{11} + 2d_{01} - d_{00}$

$= -\sum_{t=2}^{n}(y_t - y_{t-1})^2 < 0$ a.s.

$c(-1) = -(n-1)d_{11} - (n-2)d_{01} + d_{00} + nd_{11} + nd_{01}$

$= d_{11} + 2d_{01} + d_{01}$

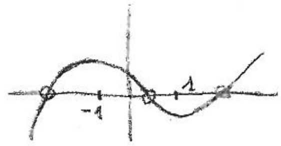
$= \sum_{t=2}^{n}(y_t + y_{t-1})^2 > 0$ a.s.

$d_{00} = \sum_{t=1}^{n} y_t^2$

$d_{01} = \sum_{t=2}^{n} y_t y_{t-1}$

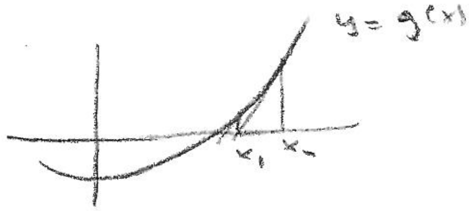
$d_{11} = \sum_{t=2}^{n-1} y_t^2$

dvs. tre reelle rødder a.s.



rod i midterste int. er $\hat{\rho}$ □

Newton-Raphson



tangent i x_0:

$$y = g(x_0) + g'(x_0)(x - x_0)$$

$$y = 0: \quad x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

fortsat: $x_{k+1} = x_k - \dfrac{g(x_k)}{g'(x_k)}$

flerdim. udgave: $x_{s+1} = x_s - \left(\dfrac{dg}{dx^T}(x_s)\right)^{-1} g(x_s)$

↑ Jacobimatricen

eks. $(y_1, \dots, y_n)$, $Y_i \sim \Gamma(\omega, \lambda)$ uafh.

$$f(y_i; \omega, \lambda) = \frac{\lambda^\omega}{\Gamma(\omega)} y_i^{\omega-1} e^{-\lambda y_i}, \quad y_i > 0$$

reparametrisering: $\mu = \dfrac{\omega}{\lambda}$, $\lambda = \dfrac{\omega}{\mu}$

$$f(y_i; \mu, \omega) = \frac{1}{\Gamma(\omega)} \left(\frac{\omega}{\mu}\right)^\omega y_i^{\omega-1} e^{-\frac{\omega}{\mu} y_i}, \quad y_i > 0$$

$$l(\omega, \lambda; y) = -n \ln \Gamma(\omega) + n\omega(\ln \omega - \ln \mu) + (\omega - 1)\sum_i \ln y_i - \frac{\omega}{\mu}\sum y_i$$

$$\frac{\partial l}{\partial \mu} = -\frac{n\omega}{\mu} + \frac{\omega}{\mu^2}\sum_i y_i = 0 \quad (1)$$

$\psi(\omega) = \dfrac{\Gamma'(\omega)}{\Gamma(\omega)}$

$$\frac{\partial l}{\partial \omega} = -n\psi(\omega) + n(\ln \omega + 1 - \ln \mu) + \sum_i \ln y_i - \frac{\sum y_i}{\mu} = 0 \quad (2)$$

↑ digammafkt.

(1) $\hat{\mu} = \bar{y}$

(2) $-n\psi(\omega) + n \ln \dfrac{\omega}{\bar{y}} + \sum_i \ln y_i = 0 \iff g(\omega) = 0$

Newton-Raphson på $g(\omega)$, se tabel i bog side 65 $\square$

ehs. $(y_1, \dots, y_n)$, $Y_i \sim e(\exp(\alpha + \beta x_i))$ uafh.

$\sum_i x_i = 0$ (ellers $\alpha + \beta x_i = \underbrace{\alpha_0}_{\text{nyt } \alpha} + \beta \underbrace{(x_i - \bar{x})}_{\text{nyt } x_i}$)

$\theta = (\alpha, \beta)$

$$L(\alpha, \beta; y) = \prod_i \exp(\alpha + \beta x_i) \exp(-y_i \exp(\alpha + \beta x_i))$$

$$\ell(\alpha, \beta; y) = \sum_i (\alpha + \beta x_i) - \sum_i y_i \exp(\alpha + \beta x_i)$$

$$= n\alpha + 0 - e^{\alpha} \sum_i y_i \exp \beta x_i$$

$$\frac{\partial \ell}{\partial \alpha} = n - e^{\alpha} \sum_i y_i \exp \beta x_i \qquad (1)$$

$$\frac{\partial \ell}{\partial \beta} = -e^{\alpha} \sum_i x_i y_i e^{\beta x_i} \qquad (2)$$

(1) $\hat{\alpha} = \ln \dfrac{n}{\sum_i y_i e^{\hat{\beta} x_i}}$

(2) $\sum_i x_i y_i e^{\hat{\beta} x_i} = 0$

$$\frac{\partial^2 \ell}{\partial \theta \partial \theta^T} = \begin{bmatrix} -e^{\alpha} \sum_i y_i e^{\beta x_i} & -e^{\alpha} \sum_i x_i y_i e^{\beta x_i} \\ -e^{\alpha} \sum_i x_i y_i e^{\beta x_i} & -e^{\alpha} \sum_i x_i^2 y_i e^{\beta x_i} \end{bmatrix}$$

$$\left| \frac{\partial^2 \ell}{\partial \theta \partial \theta^T} \right|_{\theta = \hat{\theta}} = e^{2\hat{\alpha}} \sum_i y_i e^{\hat{\beta} x_i} \sum_i x_i^2 y_i e^{\hat{\beta} x_i} > 0$$

dvs. maks. (entydig løsn. af likelihood ligninger)

Newton-Raphson, se tabel i bog side 67

$\square$

Baggrund / estimation

θ én-dim:

$$\ell(\theta) = \ell(\hat\theta) + \ell'(\hat\theta)(\theta-\hat\theta) + \tfrac{1}{2}\ell''(\hat\theta)(\theta-\hat\theta)^2 + \ldots$$

$\ell''(\hat\theta) \sim$ krumningen i $\hat\theta$, stejlhed

$J(\hat\theta) = -\ell''(\hat\theta)$ observeret information

θ fler-dim:

$$J(\hat\theta) = -\frac{d^2}{d\theta\, d\theta^T}\ell(\theta)\Big|_{\theta=\hat\theta} = -\left[\frac{\partial^2 \ell(\theta)}{\partial\theta_i\,\partial\theta_j}\Big|_{\theta=\hat\theta}\right]$$

Gentagne stikprøver

$\hat\theta$ opfattes som stok. var

Er $\hat\theta$ tæt på θ (den sande parameterværdi)?

$E[\hat\theta;\theta]-\theta$ bias

Når $E[\hat\theta;\theta]=\theta$ for alle $\theta\in\Theta$,
så er $\hat\theta$ central (unbiased)

Når $\lim_{n\to\infty} E[\hat\theta;\theta]=\theta$ for alle $\theta\in\Theta$,
så er $\hat\theta$ asymptotisk central

eks. $(y_1,\ldots,y_n)$, $Y_i \sim N(\mu,\sigma^2)$ uafh.

$\hat\mu = \bar Y$ $\hat\sigma^2 = \frac{1}{n}\sum(Y_i-\bar Y)^2$

$E[\hat\mu]=\mu$ $E[\hat\sigma^2] = \frac{n-1}{n}\sigma^2$

$\hat\mu$ er central $\hat\sigma^2$ er asymptotisk central sæt $s^2 = \frac{n}{n-1}\hat{\sigma}^2$, $E[S^2] = \sigma^2$

dvs. s^2 er central □

Regulært estimationsproblem

a statistisk model skal være identificerbar

b $\Theta \subseteq R^k$ skal være et åbent område

c alle tætheder i modellen skal have samme støtte

d diff. og int. skal kunne ombyttes

$$\int_y \frac{\partial}{\partial\theta} f(y;\theta)\,d\nu(y) = \frac{\partial}{\partial\theta}\int_y f(y;\theta)\,d\nu(y)$$

$$\int_y \frac{\partial^2}{\partial\theta^2} f(y;\theta)\,d\nu(y) = \frac{\partial^2}{\partial\theta^2}\int_y f(y;\theta)\,d\nu(y)$$

$(k=1)$

$$\int_y \frac{d^2}{d\theta\, d\theta^T} f(y;\theta)\,d\nu(y) = \frac{d^2}{d\theta\, d\theta^T}\int_y f(y;\theta)\,d\nu(y)$$

$(k>1)$

Fisher scoringsfunktion

$$u(\theta;y) = \frac{d}{d\theta}\, l(\theta;y) \qquad (\text{vektorfkt. når } k>1)$$

$$E[u(\theta;Y);\theta] = E\left[\frac{\partial}{\partial\theta}\ln f(Y;\theta);\theta\right]$$

$$= \int_y \frac{1}{f(y;\theta)}\frac{\partial}{\partial\theta} f(y;\theta)\, f(y;\theta)\,d\nu(y)$$

$$= \frac{\partial}{\partial\theta}\int_y f(y;\theta)\,d\nu(y) = \frac{\partial}{\partial\theta} 1 = 0$$

Forventet Fisher information

$$I(\theta) = Var[u(\theta;Y);\theta] = E[(u(\theta;Y))^2;\theta]$$

$k>1$: $I(\theta) = E[u(\theta;Y)\,u(\theta;Y)^T;\theta]$

$I(\theta)$ er pos. semidefinit, idet

$$\forall a: \quad a^T I(\theta)\bar{a} = \bar{a}^T E[u u^T]\bar{a}$$
$$= E[a^T u\, u^T a]$$
$$= E[\|a^T u\|^2] \geq 0$$

$k=1$:

$$I(\theta) = E[(u(\theta;Y))^2;\theta]$$
$$= \int_y \left(\frac{1}{f(y;\theta)}\frac{\partial}{\partial\theta} f(y;\theta)\right)^2 f(y;\theta)\,d\nu(y)$$

[bemærk $\frac{\partial^2}{\partial\theta^2}\ln f = \frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial f}{\partial\theta}\right)$

$$= -\frac{1}{f^2}\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\theta} + \frac{1}{f}\frac{\partial^2 f}{\partial\theta^2}$$

$$\Rightarrow \left(\frac{1}{f}\frac{\partial f}{\partial\theta}\right)^2 = \frac{1}{f}\frac{\partial^2 f}{\partial\theta^2} - \frac{\partial^2}{\partial\theta^2}\ln f$$

$$= \int_y \left(\frac{1}{f(y;\theta)}\frac{\partial^2}{\partial\theta^2} f(y;\theta) - \frac{\partial^2}{\partial\theta^2}\ln f(y;\theta)\right) f(y;\theta)\,d\nu(y)$$

$$= \int_y \left(\frac{\partial^2}{\partial\theta^2} f(y;\theta) - \frac{\partial}{\partial\theta} u(\theta;y)\, f(y;\theta)\right) d\nu(y)$$

$$= -E\left[\frac{\partial}{\partial\theta} u(\theta;y);\theta\right]$$

$k>1$:

$$I(\theta) = -E\left[\frac{d}{d\theta^T} u(\theta;Y);\theta\right]$$

(Y_1, Y_2) Y_1, Y_2 uafh. samme parameterrum

$I(\theta) = I_1(\theta) + I_2(\theta)$ ses umiddelbart

$Y_1, \dots, Y_n$ uafh., identisk fordelte:

$I(\theta) = n\, i(\theta)$, hvor $i(\theta)$ inf. for Y_i

Reparametrisering:

$\psi : \Theta \to \Psi$, $\psi = \psi(\theta)$ $(k=1)$

$$
\begin{aligned}
I_\Psi(\psi) &= E\left[(u(\psi;Y))^2 ; \psi \right] \\
&= E\left[\left(\frac{\partial}{\partial\theta} l(\theta;Y) \frac{d\theta}{d\psi} \right)^2 ; \theta \right]_{\theta=\theta(\psi)} \\
&= \left(\frac{d\theta}{d\psi} \right)^2 E\left[(u(\theta;Y))^2 ; \theta \right]_{\theta=\theta(\psi)} \\
&= \left(\frac{d\theta}{d\psi} \right)^2 I(\theta) \Big|_{\theta=\theta(\psi)}
\end{aligned}
$$

$k > 1$:

$$
I_\Psi(\psi) = \Delta^T I(\theta) \Delta \Big|_{\theta(\psi)}, \quad \text{hvor } \Delta = \left[\frac{\partial\theta_i}{\partial\psi_j} \right]
$$