Stat. 12.3.2010

Hypotesetest

Relativ likelihood

$\tilde{L}(\theta) = \dfrac{L(\theta)}{L(\hat{\theta})}$, i alm. $0 \leq \tilde{L}(\theta) \leq 1$

(egentlig en version af L)

$\tilde{L}(\theta_0)$ kan opfattes som indeks for overensstemmelse mellem data og hypotese, dvs.

$\tilde{L}(\theta_0) = 1$ perfekt overensstemmelse

$\tilde{L}(\theta_0) = 0$ ingen overensstemmelse overhovedet

Kritisk værdi

Ofte vælges en værdi r, så

$\tilde{L}(\theta_0) \leq r \sim \tilde{L}(\theta_0)$ skønnes for lav til at antage overensstemmelse mellem data og hypotese, dvs. hypotesen forkastes

$\tilde{L}(\theta_0) > r \sim$ hypotesen accepteres

Modifikationer til denne enten / eller regel ?

Statistisk test

$T : \mathcal{Y} \to \{\Theta_0, \Theta_1\}$, hvor $\Theta_0 \cup \Theta_1 = \Theta$ (parameterrummet)

ofte skrives $H_0 : \theta \in \Theta_0$ (nul hypotese)

$H_1 : \theta \in \Theta_1$ (alternativ hypotese)

Acceptområde : $\{y \in \mathcal{Y} \mid J(y) = \Theta_0\} = \mathcal{Y}_0$

Kritisk område : $\mathcal{Y} \setminus \mathcal{Y}_0 = \mathcal{Y}_1$

Teststørrelse $T(y)$ ~ reel stikprøvefkt.

fx $T(y) = \tilde{L}(\theta_0; y)$

Fejlmuligheder

$\theta \in \Theta_0 \wedge y \in \mathcal{Y}_1$: fejl af 1. art
(forkaste en sand hypotese)

$\theta \in \Theta_1 \wedge y \in \mathcal{Y}_0$: fejl af 2. art
(acceptere en falsk hypotese)

Styrkefunktion

$$\gamma(\theta) = P\left(J(y) = \Theta_1 ; \theta\right)$$

($1-\gamma(\theta)$ kaldes operationskarakteristik, OC,
benyttes især i forb. m. kvalitetskontrol)

Signifikansniveau

$$\alpha = \sup_{\theta \in \Theta_0} \gamma(\theta)$$ (= maks af
P(fejl af type I)).

eks. obs. y, $Y \sim N(\theta, 1)$

$H_0 : \theta \le 0$

$H_1 : \theta > 0$

acceptområde : $y \le \frac{1}{2}$ (et valg)

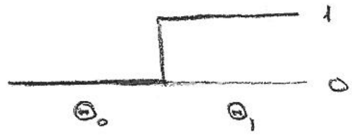
styrkefkt. : $\gamma(\theta) = P(Y > \frac{1}{2}; \theta)$

$$= P\left(\frac{Y-\theta}{1} > \frac{\frac{1}{2}-\theta}{1}; \theta\right) = \Phi(\theta - \tfrac{1}{2})$$

bemærk $\gamma(\theta)\begin{cases}\to 1 & \text{for } \theta\to\infty\\ \to 0 & \text{for } \theta\to-\infty\end{cases}$

signifikansniveau $\alpha = \gamma(0) = \Phi(-\tfrac{1}{2}) \approx 0{,}31$ □

ideel styrkefkt. (?)



men ikke opnåelig i praksis

i alm. vil forøgelse af acceptområdet

~ formindskelse af sands. for fejl af 1. art

~ forøgelse af sands. for fejl af 2. art

og omvendt

klassisk tilgang

- vælg signifikansniveau
- vælg test, så γ bliver passende stor

bemærk

H_0 og H_1 opfattes ikke symmetrisk, idet H_0 ofte specificeres i henhold til teori, erfaring o. lign., derfor ønskes normalt et lille signifikansniveau (fx 5%)

Kvotienttest

Betragt Y m. tæthed $f(y;\theta)$

$$\left.\begin{aligned} &\Theta = \{\theta_0, \theta_1\} \\ &H_0 : \theta = \theta_0 \\ &H_1 : \theta = \theta_1 \end{aligned}\right] \quad *$$

Teststørrelse

$$\lambda(y) = \frac{L(\theta_0; y)}{L(\theta_1; y)}$$

λ stor svarer til præference for θ_0

λ lille ——— " ——— θ_1

kritisk område $\mathcal{Y}_L = \{ y \mid \lambda(y) \le \lambda_\alpha \}$,

hvor $\alpha = P(\lambda(Y) \le \lambda_\alpha ; \theta_0)$, dvs.

λ_α fastlægges ud fra det valgte signifikansniveau.

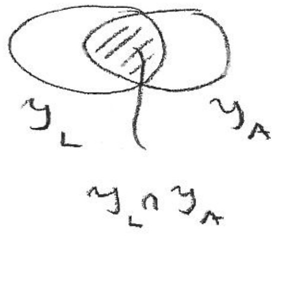
Når diskrete variable vælg største $\alpha_d \le \alpha$

Neyman-Pearsons lemma

For * (s. 3) har kvotientteststørrelsen maksimal styrke blandt alle test med signifikansniveau α.

Bevis:

Lad $\mathcal{Y}_A$ betegne et vilkårligt kritisk område med signifikansniveau $\le \alpha$



$$\alpha = \int_{\mathcal{Y}_L} f(y;\theta_0)\, d\nu(y)$$

$$\ge \int_{\mathcal{Y}_A} f(y;\theta_0)\, d\nu(y)$$

$$\Rightarrow \int_{\mathcal{Y}_L \setminus \mathcal{Y}_A} f(y;\theta_0)\, d\nu(y) \ge \int_{\mathcal{Y}_A \setminus \mathcal{Y}_L} f(y;\theta_0)\, d\nu(y)$$

da $\int_{\mathcal{Y}_L \cap \mathcal{Y}_A} f(y;\theta_0)\, d\nu(y)$ er fratrukket på begge sider

$$y \in \mathcal{Y}_L \setminus \mathcal{Y}_A \Rightarrow y \in \mathcal{Y}_L \Rightarrow \lambda_\alpha f(y;\theta_1) \geq f(y;\theta_0)$$

$$y \in \mathcal{Y}_A \setminus \mathcal{Y}_L \Rightarrow f(y;\theta_0) \geq \lambda_\alpha f(y;\theta_1)$$

dvs.

$$\lambda_\alpha \int_{\mathcal{Y}_L \setminus \mathcal{Y}_A} f(y;\theta_1)\, d\nu(y) \geq \int_{\mathcal{Y}_L \setminus \mathcal{Y}_A} f(y;\theta_0)\, d\nu(y)$$

$$\geq \int_{\mathcal{Y}_A \setminus \mathcal{Y}_L} f(y;\theta_0)\, d\nu(y) \geq \lambda_\alpha \int_{\mathcal{Y}_A \setminus \mathcal{Y}_L} f(y;\theta_1)\, d\nu(y)$$

$$\Rightarrow \int_{\mathcal{Y}_L} f(y;\theta_1)\, d\nu(y) \geq \int_{\mathcal{Y}_A} f(y;\theta_1)\, d\nu(y)$$

Teststørrelse, når

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

Kvotientteststørrelsen defineres nu som

$$\lambda(y) = \frac{L(\theta_0;y)}{\sup_{\theta \neq \theta_0} L(\theta;y)} = \frac{L(\theta_0;y)}{\sup_{\theta \in \Theta} L(\theta;y)} = \frac{L(\theta_0;y)}{L(\hat{\theta};y)}$$

dvs. den relative likelihood

Den monotone transformation $-2\ln(\cdot)$:

$$W(y) = -2\ln(\lambda(y)) = -2\left(l(\theta_0;y) - l(\hat{\theta};y)\right)$$

Bemærk små værdier af $\lambda(y)$ svarer til store værdier af $W(y)$

Andre teststørrelser

I $W(y)$ udvikles $l(\theta_0)$ i et Taylorpolynomium med $\hat{\theta}$ som udviklingspunkt:

$$W(y) = -2\left(l(\hat\theta) + l'(\hat\theta)(\theta_0-\hat\theta) + \tfrac{1}{2} l''(\tilde\theta)(\theta_0-\hat\theta)^2 - l(\hat\theta)\right)$$

$$= -(\theta_0-\hat\theta)^2 l''(\tilde\theta), \quad \tilde\theta \text{ ml. } \hat\theta \text{ og } \theta_0$$

$$= -n(\theta_0-\hat\theta)^2 \frac{l''(\theta_0)}{n} + o_p(1)$$ $\hat\theta$ konsistent $\Rightarrow \tilde\theta$ konsistent

$$= n(\hat\theta-\theta_0)^2\left(i(\theta_0)+o_p(1)\right)+o_p(1)$$ store tals lov

$$= n(\hat\theta-\theta_0)^2\left(i(\hat\theta)+o_p(1)\right)+o_p(1)$$ $i(\theta)$ kont.

$$= n(\hat\theta-\theta_0)^2 i(\hat\theta) + o_p(1) \qquad *$$

$$= W_e(y) + o_p(1)$$

$W_e(y)$ kaldes Walds teststørrelse

tidl.

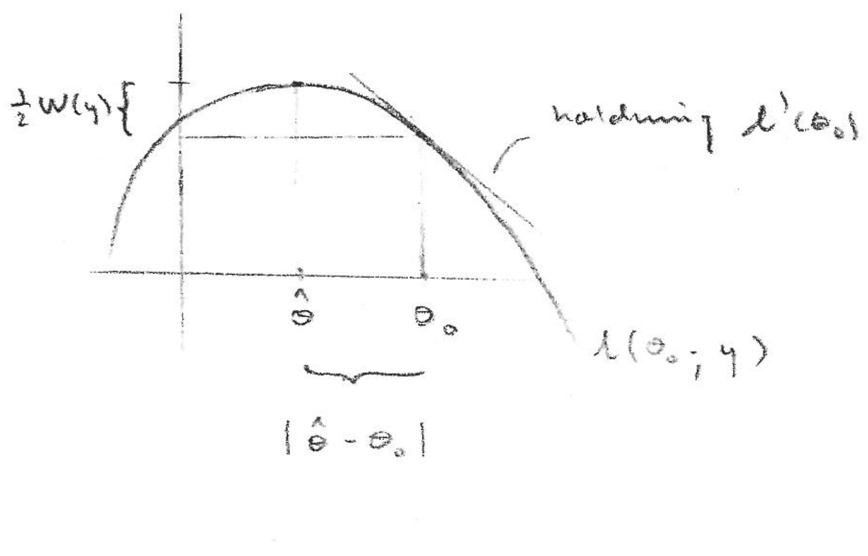
$$\sqrt{n}(\hat\theta-\theta_0) = \frac{-\frac{1}{\sqrt n} l'(\theta_0)}{\frac{1}{n} l''(\theta_0) + o_p(1)} \simeq \frac{l'(\theta_0)}{\sqrt n\left(i(\theta_0)+o_p(1)\right)}$$

$$\simeq \frac{l'(\theta)}{\sqrt n\, i(\theta_0)}$$

$$\Downarrow$$

$$n(\hat\theta-\theta_0)^2 \simeq \frac{(l'(\theta_0))^2}{n(i(\theta_0))^2}$$ indsættes i *

$$W_u(y) = \frac{(l'(\theta_0))^2}{n\, i(\theta_0)}$$ scorings teststørrelsen



eks. $y = (y_1, \dots, y_n)$, $Y_i \sim N(\theta, \sigma^2)$ uafh.

$H_0 : \theta = \theta_0$ $\qquad \ell(\theta; y) = -\frac{1}{2}\sum_i \frac{(y_i-\theta)^2}{\sigma^2}$

$H_1 : \theta \neq \theta_0$

$$W(y) = -2\left(\ell(\theta_0;y) - \ell(\hat\theta;y)\right)$$
$$= \sum_i \frac{(y_i-\theta_0)^2}{\sigma^2} - \sum_i \frac{(y_i-\hat\theta)^2}{\sigma^2}$$
$$= \frac{1}{\sigma^2}\left(\sum_i \left((y_i-\hat\theta)+(\hat\theta-\theta_0)\right)^2 - \sum_i (y_i-\hat\theta)^2\right)$$
$$= \frac{n(\hat\theta-\theta_0)^2}{\sigma^2}$$

$$W_e(y) = (\hat\theta-\theta_0)^2 I(\hat\theta) = (\hat\theta-\theta_0)^2 \frac{n}{\sigma^2}$$
$$= \frac{n(\hat\theta-\theta_0)^2}{\sigma^2}$$

$$W_u(y) = \frac{(\ell'(\theta_0))^2}{I(\theta_0)} = \frac{\left(\frac{n(\hat\theta-\theta_0)}{\sigma^2}\right)^2}{\frac{n}{\sigma^2}}$$
$$= \frac{n(\hat\theta-\theta_0)^2}{\sigma^2}$$

her $W(y) = W_e(y) = W_u(y)$

kritisk værdi

$$P\Big(\underbrace{\frac{n(\hat\theta-\theta_0)^2}{\sigma^2}}_{\substack{\sim \chi^2(1) \\ \text{under } H_0}} \geq \underbrace{-2\ln\lambda_\alpha}_{\substack{\text{må sættes til} \\ (1-\alpha)\text{-fraktilen i } \chi^2(1)}} ; \theta_0\Big) = \alpha$$

bemærk, at acceptomr. kan angives

$$\Big\{ y \,\Big|\, -z_{1-\frac{\alpha}{2}} < \underbrace{\frac{\hat\theta-\theta_0}{\frac{\sigma}{\sqrt{n}}}}_{\sim N(0,1) \text{ under } H_0} < z_{1-\frac{\alpha}{2}} \Big\}$$

$$\frac{\hat\theta-\theta_0}{\frac{\sigma}{\sqrt{n}}} = \frac{\hat\theta-\theta+\theta-\theta_0}{\frac{\sigma}{\sqrt{n}}} = \frac{\hat\theta-\theta}{\frac{\sigma}{\sqrt{n}}} + \frac{\theta-\theta_0}{\frac{\sigma}{\sqrt{n}}}$$

$$\frac{\hat{\theta}-\theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \text{ under } H_1$$

$$\delta = \frac{\theta-\theta_0}{\frac{\sigma}{\sqrt{n}}}$$ er ikke-centralitetsparametren

styrkefkt.

$$\gamma(\theta) = 1 - \Phi(z_{1-\frac{\alpha}{2}} - \delta) + \Phi(-z_{1-\frac{\alpha}{2}} - \delta)$$

$$= 2 - \Phi(z_{1-\frac{\alpha}{2}} - \delta) - \Phi(z_{1-\frac{\alpha}{2}} + \delta)$$

$\square$

Kvotienttestet generelt

teststørrelse : $$\lambda(y) = \frac{\sup_{\theta\in\Theta_0} L(\theta;y)}{\sup_{\theta\in\Theta} L(\theta;y)}$$

kritisk værdi λ_α bestemmes, så

$$\sup_{\theta\in\Theta_0} P(\lambda(Y) \le \lambda_\alpha) = \alpha$$

kritisk omr. : $R = \{ y \mid \lambda(y) < \lambda_\alpha \}$

Asymptotisk fordeling

Idet $\sqrt{n}(\hat{\theta}-\theta_0) \xrightarrow{d} N(0, \frac{1}{i(\theta_0)})$, har vi

$$W(Y) \sim \chi^2(1) \text{ asymptotisk under } H_0$$

Dermed også $\begin{cases} W_e(Y) \sim \chi^2(1) \text{ asympt. u. } H_0 \\ W_u(Y) \sim \chi^2(1) \text{ asympt. u. } H_0 \end{cases}$

Ved brug af asymptotisk test udgør forskellen mellem aktuelt niveau og nominelt niveau et problem.

Konservativt test, når

aktuelt niveau $\le$ nominelt niveau

Asymptotisk ford., når $k > 1$

$$W(y) = n(\hat{\theta} - \theta_0)^T\, i(\theta_0)\, (\hat{\theta} - \theta_0)$$

$\sim \chi^2(k)$ under H_0 asymptotisk

$$W_e(y) = n(\hat{\theta} - \theta_0)^T\, i(\hat{\theta})\, (\hat{\theta} - \theta_0)$$

$\sim \chi^2(k)$ under H_0 asymptotisk

$$W_u(y) = \left(\frac{dl}{d\theta}\Big|_{\theta=\theta_0} \right)^T \left(I(\theta_0) \right)^{-1} \left(\frac{dl}{d\theta}\Big|_{\theta=\theta_0} \right)$$

$\sim \chi^2(k)$ under H_0 asymptotisk

eks.

$y = (y_1, \dots, y_n)$, $Y_i : g(t;\theta) = \begin{cases} e^{-(t-\theta)} & , t > \theta \\ 0 & \text{ellers} \end{cases}$

$i = 1, \dots, n$ uafh.

$H_0 : \theta = \theta_0$

$H_1 : \theta > \theta_0$

NB! Ikke et regulært estimationsproblem

kvotienttest

$$\lambda(y) = \prod_i \frac{e^{-(y_i - \theta_0)}}{e^{-(y_i - \hat{\theta})}} = e^{-n(\hat{\theta} - \theta_0)}$$

$$W(y) = 2n(\hat{\theta} - \theta_0), \quad \hat{\theta} = y_{(1)}$$

$\hat{\theta} - \theta_0 \sim e(n \cdot 1) \Rightarrow W(y) \sim e(\tfrac{1}{2}) = \chi^2(2)$

under H_0

(ikke $\chi^2(1)$) $\square$

Observeret signifikansniveau (p-værdi)

$$\alpha_{obs} = \sup_{\theta \in \Theta_0} P(W(Y) \geq W(y); \theta)$$

i alm.: når $\alpha_{obs} < 0{,}05$ forkastes H_0

$\alpha_{obs} > 0{,}10$ ingen rimelig grund til at forkaste H_0

Profil likelihood

Når $\theta = (\psi, \omega)$ og nullhypotesen kun omhandler ψ (ω er 'nuisance' parameter)

$l^*(\psi) = l(\psi, \hat{\omega}_\psi)$, maks for $\psi = \hat{\psi}$, den sædvanlige MLE

Test $H_0 : \psi = \psi_0$

$H_1 : \psi \neq \psi_0$

$$w(y) = 2\,(l(\hat{\psi}, \hat{\omega}) - l(\psi_0, \hat{\omega}_{\psi_0}))$$
$$= 2\,(l^*(\hat{\psi}) - l^*(\psi_0))$$
$$\sim \chi^2(q) \text{ under } H_0, \text{ hvor}$$
$$q = \dim(\psi)$$

Når $k - q$ er høj (dvs. mange 'nuisance' parametre) er den asymptotiske ford. ikke så god.

To-sidet t-test

$y = (y_1, \dots, y_n)$, $Y_i \sim N(\mu, \sigma^2)$ uafh.

$H_0 : \mu = \mu_0$, $\Theta_0 : \sigma^2 > 0$

$H_1 : \mu \neq \mu_0$, $\Theta_1 : \mu \in R \setminus \{\mu_0\}, \sigma^2 > 0$

MLE: $\hat{\mu} = \bar{y}$, $\hat{\sigma}^2 = \frac{1}{n} \sum (y_i - \bar{y})^2$ (tidl. bestemt)

under H_0: $\hat{\sigma}_0^2 = \frac{1}{n} \sum (y_i - \mu_0)^2$

$$\lambda(y) = \frac{L(\mu_0, \hat{\sigma}_0^2)}{L(\hat{\mu}, \hat{\sigma}^2)} = \frac{(2\pi e \hat{\sigma}_0^2)^{-\frac{n}{2}}}{(2\pi e \hat{\sigma}^2)^{-\frac{n}{2}}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-\frac{n}{2}}$$

$$= \left(\frac{\frac{1}{n}\sum_i (y_i - \mu_0)^2}{\frac{1}{n}\sum_i (y_i - \bar{y})^2}\right)^{-\frac{n}{2}}$$

$$= \left(\frac{\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu_0)^2}{\sum_i (y_i - \bar{y})^2}\right)^{-\frac{n}{2}}$$

$$= \left(1 + \frac{t^2}{n-1}\right)^{-\frac{n}{2}}, \quad \text{hvor}$$

$$t = \frac{\frac{\bar{y} - \mu_0}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{\sum_i (y_i - \bar{y})^2}{\sigma^2}}{n-1}}} = \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}} \quad *$$

$T \sim t(n-1)$ under H_0

kritisk omr. $(-\infty; -t_{1-\frac{\alpha}{2}}) \cup (t_{1-\frac{\alpha}{2}}, \infty)$

p-værdi: $2\,P(T > |t_{obs}|)$

under H_1 er T ikke-centralt t-fordelt

$$T \sim t\left(n-1; \frac{\mu - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right)$$

↑ ikke-centralitetsparameter

$$\left(\begin{array}{l} \text{asymptotisk er } T^2 \sim \chi^2(1) \\ -2\ln\lambda = n\ln\left(1 + \frac{t^2}{n-1}\right) \approx n\frac{t^2}{n-1} \approx t^2 \end{array}\right)$$

* $s^2 = \frac{1}{n-1}\sum_i (y_i - \bar{y})^2$

2.9 Non-central distributions

We now consider the non-central χ^2, t and F-distributions, and show how they may be used to obtain the power functions of the t- and F-tests.

2.9.1 The non-central χ^2-distribution

Let $Y_1, \ldots, Y_n$ be independent, with distributions

$$Y_i \sim N(\xi_i, 1), \qquad i = 1, \ldots, n,$$

where $\xi = (\xi_1, \ldots, \xi_n)^T \in \mathbf{R}^n$. We study the distribution of

$$U = \sum_{i=1}^{n} Y_i^2 .$$

If $\xi = 0$, the distribution of U is a standard (central) χ^2-distribution. For $\xi \neq 0$, the distribution of U is called a *non-central χ^2-distribution*, with n degrees of freedom and non-centrality parameter $\delta = ||\xi||$, and is denoted

$$U \sim \chi^2(n; \delta).$$

In particular, the central χ^2-distribution is $\chi^2(n) = \chi^2(n; 0)$.

Let us verify that the distribution of U depends on ξ only through $||\xi||$. Let $Y = (Y_1, \ldots, Y_n)^T$, so that $U = ||Y||^2$. Let us decompose Y as

$$Y = \xi + \varepsilon,$$

where the components of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ have distributions $\varepsilon_i \sim N(0, 1)$, $i = 1, \ldots, n$. Let A be an orthogonal matrix. Then

$$||AY||^2 = Y^T A^T A Y = Y^T Y = ||Y||^2 ,$$

by the orthogonality of the columns of A. Hence, we find that

$$P(U > u) = P(||Y||^2 > u) = P(||AY||^2 > u) = P(||A\xi + A\varepsilon||^2 > u).$$

By Theorem 2, the variable $A\varepsilon$ has the same distribution as ε, so

$$P(U > u) = P(||A\xi + \varepsilon||^2 > u).$$

We may choose A such that, for $\delta = \|\xi\|$ we have $A\xi = (\delta, 0, \dots, 0)^T$. It follows that $P(U > u)$ depends on ξ only through $\delta = \|\xi\|$, as we had to show. The result is illustrated geometrically in Figure 9.1.

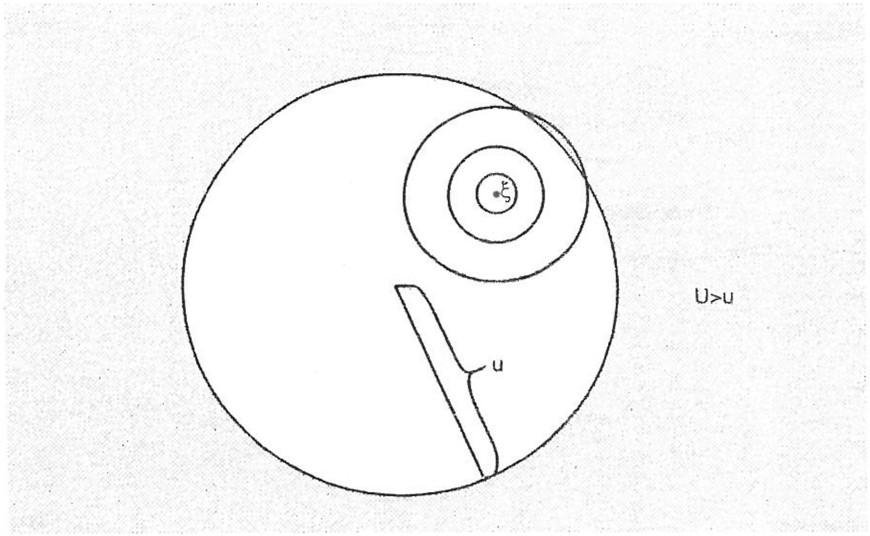


Figure 9.1 *Contours of the density function of Y (small circles) and the event $\{U > u\}$ (hatched)*

The figure illustrates the fact that the contours for the density function of Y are circles with center ξ. The probability $P(U > u)$ is invariant under rotations, and hence depends on ξ only through $\|\xi\|$.

According to the proof, we may take a new coordinate system defined by

$$e_1 = \frac{\xi}{\|\xi\|}$$

and $e_2, \dots, e_n$ orthogonal to e_1, such that $e_1, \dots, e_n$ is an orthonormal basis for $\mathbf{R}^n$. Let $X = (X_1, \dots, X_n)$ denote the coordinates of Y in the basis $(e_1, \dots, e_n)$. Then $X_i = e_i \cdot Y$ and

$$\mathrm{E}X_1 = \frac{\xi}{\|\xi\|} \cdot \xi = \|\xi\| = \delta$$

$$\mathrm{E}X_i = e_i \cdot \xi = 0, \qquad i = 2, \dots, n.$$

Now

$$U = \|Y\|^2 = \|X\|^2 = X_1^2 + \sum_{i=2}^n X_i^2$$

$$= (\delta + \varphi_1)^2 + \sum_{i=2}^n \varphi_i^2, \tag{9.1}$$

where $\varphi_1, \dots, \varphi_n$ are independent, with distributions $N(0,1)$. Hence, we have decomposed U into the sum of a $\chi^2(1;\delta)$-variable and an independent $\chi^2(n-1)$-variable.

We obtain

$$\mathrm{E}[(\varphi_1 + \delta)^2] = \mathrm{Var}(\delta + \varphi_1) + \{\mathrm{E}(\delta + \varphi_1)\}^2 = 1 + \delta^2.$$

Using $\mathrm{E}\{\chi^2(n-1)\} = n - 1$, we hence obtain

$$\mathrm{E}U = \delta^2 + n. \tag{9.2}$$

Using formulas for the fourth moment of the normal distribution, we obtain

$$\mathrm{Var}\ U = 4\delta^2 + 2n. \tag{9.3}$$

Finally, we state a convolution formula for the non-central χ^2-distribution. Let U_1 and U_2 be independent, $U_i \sim \chi^2(n_i; \delta_i)$, $i = 1, 2$. Then

$$U_1 + U_2 \sim \chi^2(n_1 + n_2; (\delta_1^2 + \delta_2^2)^{\frac{1}{2}}). \tag{9.4}$$

To show (9.4), we represent U_1 and U_2 in $\mathbf{R}^n$, $n = n_1 + n_2$. Let $Y_1, Y_2 \in \mathbf{R}^n$ be random vectors, such that

$$Y_1 = \begin{pmatrix} \delta_1 + \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{n_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad Y_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta_2 + \varepsilon_{n_1+1} \\ \varepsilon_{n_1+2} \\ \vdots \\ \varepsilon_n \end{pmatrix}.$$

We may write $U_1 = \|Y_1\|^2$ and $U_2 = \|Y_2\|^2$, due to (9.1), where $\varepsilon_1, \dots, \varepsilon_n$ are independent and identically distributed $N(0,1)$. Due to the independence of U_1 and U_2, we may in fact choose $(\varepsilon_1, \dots, \varepsilon_{n_1})$ independent of $(\varepsilon_{n_1+1}, \dots, \varepsilon_n)$. Now

$$U_1 + U_2 = \|Y_1\|^2 + \|Y_2\|^2 = \|Y_1 + Y_2\|^2$$

by Pythagoras' Theorem, because $Y_1 \perp Y_2$. Furthermore

$$Y_1 + Y_2 = \xi + \varepsilon,$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T$ and

$$\xi = (\delta_1, 0, \dots, 0, \delta_2, 0, \dots, 0)^T.$$

Hence $U_1 + U_2$ is non-central χ^2-distributed with degrees of freedom n and non-centrality parameter

$$\|\xi\| = (\delta_1^2 + \delta_2^2)^{\frac{1}{2}}.$$

2.9.2 The non-central F- and t-distributions

Let U_1 and U_2 be independent,

$$U_1 \sim \chi^2(f_1; \delta) \qquad U_2 \sim \chi^2(f_2).$$

Then the distribution of

$$F = \frac{U_1/f_1}{U_2/f_2}$$

is called a *non-central F-distribution*

$$F \sim F(f_1, f_2; \delta)$$

with (f_1, f_2) degrees of freedom, and non-centrality parameter δ.

Similarly, if Y and U are independent, and

$$Y \sim N(\delta, 1) \qquad U \sim \chi^2(f),$$

then the distribution of

$$t = \frac{Y}{\sqrt{U/f}}$$

is called a *non-central t-distribution*,

$$t \sim t(f; \delta),$$

with f degrees of freedom and non-centrality parameter δ. Note that, generalizing the result (6.7) in the central case, we have

$$t^2(f; \delta) = F(1, f; \delta).$$

2.9.3 The power of F- and t-tests

Consider the standard linear model where $Y_1, \dots, Y_n$ are independent,

$$Y_i = \mu_i + \varepsilon_i$$

with $\varepsilon_i \sim N(0,1)$, $\mu = (\mu_1, \dots, \mu_n)^T$, $Y = (Y_1, \dots, Y_n)^T$. Let $L_1 \subseteq L_2$ be two linear hypotheses, and consider the F-statistic for L_2 under L_1,

$$F(y) = \frac{\|p_1(y) - p_2(y)\|^2 / (k_1 - k_2)}{\|y - p_1(y)\|^2 / (n - k_1)}.$$

Under the hypothesis L_2, that is, for $\mu \in L_2$, we know that F has a central F-distribution

$$F(Y) \sim F(k_1 - k_2, n - k_1).$$

In particular, the distribution of $F(Y)$ does not depend on the value of μ, as long as $\mu \in L_2$.

Now, consider the distribution of $F(Y)$ for $\mu \in L_1 \setminus L_2$. First note that

$$Y - p_1(Y) = \mu + \varepsilon - p_1(\mu) - p_1(\varepsilon) = \varepsilon - p_1(\varepsilon),$$

where we have used $p_1(\mu) = \mu$, which follows from $\mu \in L_1 \setminus L_2 \subseteq L_1$. Hence

$$\|Y - p_1(Y)\|^2 = \|\varepsilon - p_1(\varepsilon)\|^2 \sim \sigma^2 \chi^2(n - k_1), \tag{9.5}$$

according to our earlier results.

For the vector $p_1(Y) - p_2(Y)$, the result is different. We have

$$p_1(Y) - p_2(Y) = p_1(\mu) + p_1(\varepsilon) - p_2(\mu) - p_2(\varepsilon).$$

As before, $p_1(\mu) = \mu$, because $\mu \in L_1$. Since $\mu \notin L_2$ we find $p_2(\mu) \neq \mu$ so that

$$p_1(Y) - p_2(Y) = p_1(\varepsilon) - p_2(\varepsilon) + \mu - p_2(\mu). \tag{9.6}$$

Now, since $p_1(\varepsilon) - p_2(\varepsilon)$ is independent of $\varepsilon - p_1(\varepsilon)$, as shown earlier, we find that $p_1(Y) - p_2(Y)$ and $Y - p_1(Y)$ are independent.

Now, by (9.5), we have

$$\frac{1}{\sigma^2} \|Y - p_1(Y)\|^2 \sim \chi^2(n - k_1), \tag{9.7}$$

and using (9.6)

$$\begin{aligned} \frac{1}{\sigma^2} \|p_1(Y) - p_2(Y)\|^2 &= \left\| \frac{1}{\sigma} \{p_1(Y) - p_2(Y)\} \right\|^2 \\ &= \left\| p_1(\frac{\varepsilon}{\sigma}) - p_2(\frac{\varepsilon}{\sigma}) + \frac{1}{\sigma}\{\mu - p_2(\mu)\} \right\|^2. \end{aligned}$$

We may now show, since $\varepsilon_i/\sigma \sim N(0,1)$, that

$$\frac{1}{\sigma^2}\|p_1(Y)-p_2(Y)\|^2 \sim \chi^2(k_1-k_2;\delta) \tag{9.8}$$

where

$$\delta = \left\|\frac{1}{\sigma}\{\mu - p_2(\mu)\}\right\| = \frac{1}{\sigma}\|\mu - p_2(\mu)\|.$$

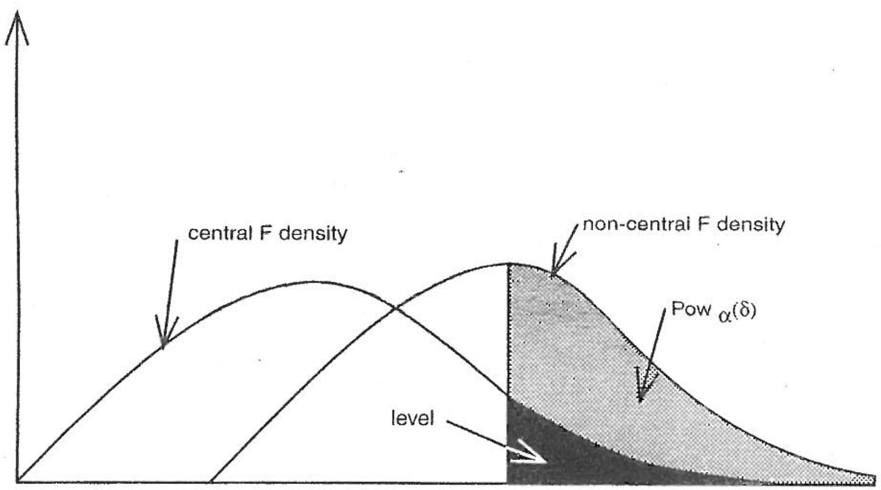


Figure 9.2 *Power of the F-test (hatched and black) and level (black)*

Note that to prove (9.8), we have to represent $p_1(\varepsilon/\sigma)-p_2(\varepsilon/\sigma)$ as a vector in $\mathbf{R}^{k_1-k_2}$ with independent and identically distributed $N(0,1)$ components, cf. Exercise 2.9. The non-centrality parameter hence measures, in units of σ, the orthogonal distance between μ and L_2.

By the definition of the non-central F-distribution, we hence obtain

$$F(Y) \sim F(k_1-k_2, n-k_1;\delta). \tag{9.9}$$

We may now calculate the power of the F-test. Let α be the level of the F-test, so that we reject L_2 if

$$F(y) > F_{1-\alpha}(k_1-k_2, n-k_1).$$

The power of the F-test is hence a function of δ alone,

$$\mathrm{Pow}_\alpha(\delta) = P(F(Y) > F_{1-\alpha}(k_1-k_2, n-k_1)),$$

where $F(Y)$ has distribution (9.9). Figure 9.2 illustrates the relationship between the level and the power of the F-test.

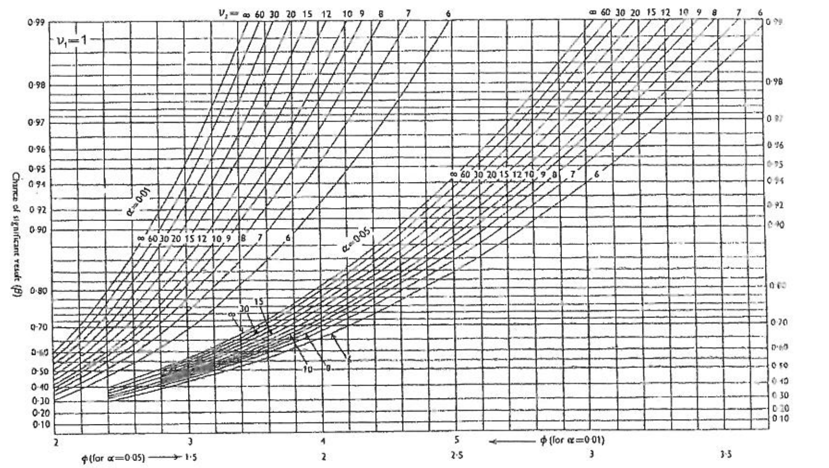


Figure 9.3 *Pearson and Hartley charts for the power of the F-test*[1]

As Figure 9.2 illustrates, the non-central F-distribution is shifted to the right, compared with a central F-distribution. To verify this, note that from (9.7) that

$$\mathrm{E}\left\{\frac{\|Y-p_1(Y)\|^2}{n-k_1}\right\} = \sigma^2$$

for $\mu \in L_2$ and also for $\mu \in L_1 \setminus L_2$. Now, from (9.7) we find, using (9.2)

$$\mathrm{E}\{\|p_1(Y)-p_2(Y)\|^2/(k_1-k_2)\} = \sigma^2\left(1+\frac{\delta^2}{k_1-k_2}\right),$$

which confirms that the mean of the numerator of the F-statistic is an increasing function of δ. For $\delta = 0$, F is the ratio of two independent quantities, both with mean σ^2, so under L_2 we expect F to be near 1.

[1] From *Biometrika Tables for Statisticians* Vol. II by E.S. Pearson and H.O. Hartley p. 250.

Figure 9.3 shows a graphical display to calculate the power of the F-test. The degrees of freedom are denoted ν_1 and ν_2, ϕ is the non-centrality parameter, $\alpha(= 0.01$ or $0.05)$ is the level, and β is the power. The graph is for the cases $\nu_1 = 1$ or 2. Further graphs for higher values of ν_1 may be found in Pearson and Hartley (1972, pp 250–259).

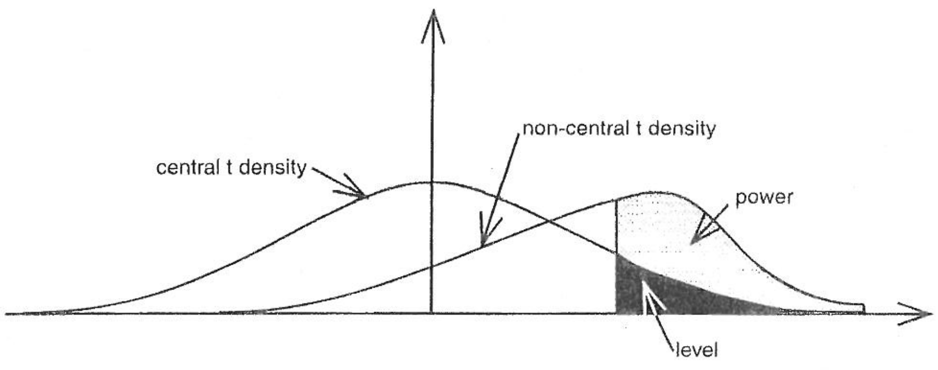


Figure 9.4 *Power (hatched and black) and level (black) for a one-sided t-test*

The power of the two-sided t-test may be derived from the power of the F-test, because $t^2(f) = F(1, f)$. Now we consider the power of the one-sided t-test. Following the notation of Section 2.6 and 2.7, we test the hypothesis $\beta_k = \beta_0$, using the t-statistic

$$t(y) = \frac{\hat{\beta}_k - \beta_0}{\text{s.e.}(\hat{\beta}_k)}.$$

If the true value β of β_k is different from β_0, we have

$$\hat{\beta}_k - \beta_0 \sim N(\beta - \beta_0, \sigma^2 c_{jj}).$$

Hence, since $\hat{\beta}_k$ and s.e.$(\hat{\beta}_k)$ are independent, the distribution of $t(Y)$ is

$$t(Y) \sim t\left(n - k; \frac{\beta - \beta_0}{\sigma c_{jj}^{1/2}}\right). \qquad (9.10)$$

Suppose we make a test for $\beta_k = \beta_0$, with alternative hypothesis $H_A : \beta_k > \beta_0$. Then the t-test will reject the hypothesis for

$$t(y) > t_{1-\alpha}(n - k),$$

The power of the test is hence

$$P(t(Y) > t_{1-\alpha}(n - k)),$$

where $t(Y)$ has distribution (9.10). Figure 9.4 illustrates the situation.