En-sidet t-test

$y = (y_1, \dots, y_n)$, $Y_i \sim N(\mu, \sigma^2)$ uafh.

$H_0 : \mu \le \mu_0 \qquad \Theta_0 : \mu \le \mu_0 \, , \sigma^2 > 0$

$H_1 : \mu > \mu_0 \qquad \Theta_1 : \mu > \mu_0 \, , \sigma^2 > 0$ (opad ensidet)

MLE: $\hat\mu = \bar y$, $\hat\sigma^2 = \frac{1}{n} \Sigma (y_i - \bar y)^2$

under H_0:

$$\hat\mu_0 = \min \{ \bar y, \mu_0 \} , \quad \hat\sigma_0^2 = \frac{1}{n} \Sigma (y_i - \hat\mu_0)^2$$

$$\lambda(y) = \left(\frac{\hat\sigma_0^2}{\hat\sigma^2} \right)^{-\frac{n}{2}} = \left(\frac{\sum_i (y_i - \hat\mu_0)^2}{\sum_i (y_i - \bar y)^2} \right)^{-\frac{n}{2}} = \begin{cases} 1 & \text{for } \hat\mu_0 = \bar y \\ < 1 & \text{for } \hat\mu_0 = \mu_0 \end{cases}$$

$\lambda(y) = 1$: H_0 accepteres

$\lambda(y) < 1$:

$$\lambda(y) = \left(1 + \frac{t^2}{n-1} \right)^{-\frac{n}{2}} , \qquad t = \frac{\bar y - \mu_0}{\frac{s}{\sqrt n}}$$

kritisk omr.: $t^2 > t^2_{1-\alpha} \wedge \bar y > \mu_0$

$$\Leftrightarrow \quad t > t_{1-\alpha}(n-1)$$

p-værdi: $P(T > t_{obs})$

En-sidet test opfylder ikke forudsætningerne for at få en χ^2-ford. som asymptotisk ford.

Her: $\lambda(Y)$ ækvivalent med $T \sim t(n-1)$

Bemærk $T \xrightarrow{d} U \sim N(0,1)$ for $n \to \infty$

To uafh. obs. rækker, t-test for ens middelværdier

$$\left.\begin{array}{ll} z = (z_1, \ldots, z_n), & Z_i \sim N(\mu, \sigma^2) \\ x = (x_1, \ldots, x_m), & X_j \sim N(\eta, \sigma^2) \end{array}\right\} \text{uafh.}$$

$H_0: \mu = \eta$

$H_1: \mu \neq \eta$ (altså tosidet)

$$L(\mu, \eta, \sigma^2) = c\, \sigma^{-(n+m)} \exp\left(-\frac{1}{2\sigma^2}\left(\sum_i (z_i - \mu)^2 + \sum_j (x_j - \eta)^2\right)\right)$$

$$l(\mu, \eta, \sigma^2) = -\frac{n+m}{2} \ln \sigma^2 - \frac{1}{2\sigma^2}\left(\sum_i (z_i - \mu)^2 + \sum_j (x_j - \eta)^2\right)$$

heraf MLE: $\hat{\mu} = \bar{z}$, $\hat{\eta} = \bar{x}$

$$\hat{\sigma}^2 = \frac{1}{n+m}\left(\sum_i (z_i - \bar{z})^2 + \sum_j (x_j - \bar{x})^2\right)$$

under H_0:

$$l(\mu_0, \sigma_0^2) = -\frac{n+m}{2} \ln \sigma_0^2 - \frac{1}{2\sigma_0^2}\left(\sum_i (z_i - \mu_0)^2 + \sum_j (x_j - \mu_0)^2\right) \quad *$$

** $g(\mu_0) = n(\bar{z} - \mu_0)^2 + m(\bar{x} - \mu_0)^2$

$g'(\mu_0) = -2n(\bar{z} - \mu_0) - 2m(\bar{x} - \mu_0)$

$= 0$ for $(n+m)\mu_0 = n\bar{z} + m\bar{x}$

$\longrightarrow$ heraf $\hat{\mu}_0 = \dfrac{n\bar{z} + m\bar{x}}{n+m}$

$$\hat{\sigma}_0^2 = \frac{1}{n+m}\left(\sum_i (z_i - \hat{\mu}_0)^2 + \sum_j (x_j - \hat{\mu}_0)^2\right)$$

$$\lambda(z, x) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-\frac{n+m}{2}}$$

$$= \left(\frac{\sum_i (z_i - \hat{\mu}_0)^2 + \sum_j (x_j - \hat{\mu}_0)^2}{\sum_i (z_i - \bar{z})^2 + \sum_j (x_j - \bar{x})^2}\right)^{-\frac{n+m}{2}}$$

$$= \left(\frac{\sum_i (z_i - \bar{z})^2 + n(\bar{z} - \hat{\mu}_0)^2 + \sum_j (x_j - \bar{x})^2 + m(\bar{x} - \hat{\mu}_0)^2}{\sum_i (z_i - \bar{z})^2 + \sum_j (x_j - \bar{x})^2}\right)^{-\frac{n+m}{2}}$$

$*$ $\sum_i (z_i - \mu_0)^2 + \sum_j (x_j - \mu_0)^2 = \sum_i ((z_i - \bar{z}) + (\bar{z} - \mu_0))^2 + \sum_j ((x_j - \bar{x}) + (\bar{x} - \mu_0))^2$

$= \sum_i (z_i - \bar{z})^2 + n(\bar{z} - \mu_0)^2 + \sum_j (x_j - \bar{x})^2 + m(\bar{x} - \mu_0)^2 \quad **$

bemærk $n(\bar z - \hat\mu_0)^2 + m(\bar x - \hat\mu_0)^2$

$$= n\left(\bar z - \frac{n\bar z + m\bar x}{n+m}\right)^2 + m\left(\bar x - \frac{n\bar z - m\bar x}{n+m}\right)^2$$

$$= \frac{n}{(n+m)^2}\left(n\bar z + m\bar z - n\bar z - m\bar x\right)^2 + \frac{m}{(n+m)^2}\left(n\bar x + m\bar x - n\bar z - m\bar x\right)^2$$

$$= \frac{nm^2}{(n+m)^2}(\bar z - \bar x)^2 + \frac{mn^2}{(n+m)^2}(\bar x - \bar z)^2$$

$$= \frac{nm}{n+m}(\bar z - \bar x)^2 \quad ; \quad \frac{nm}{n+m} = \left(\frac{1}{n} + \frac{1}{m}\right)^{-1}$$

$$\lambda(z,x) = \left(1 + \frac{\dfrac{(\bar z - \bar x)^2}{\frac{1}{n} + \frac{1}{m}}}{\sum_i (z_i - \bar z)^2 + \sum_j (x_j - \bar x)^2}\right)^{-\frac{n+m}{2}}$$

$$= \left(1 + \frac{t^2}{n+m-2}\right)^{-\frac{n+m}{2}},$$ hvor

$$t = \frac{\bar z - \bar x}{s\sqrt{\frac{1}{n} + \frac{1}{m}}} = \frac{\dfrac{\bar z - \bar x}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\dfrac{\frac{(n+m-2)s^2}{\sigma^2}}{n+m-2}}} \quad \text{og} \quad s^2 = \frac{\sum_i (z_i - \bar z)^2 + \sum_j (x_j - \bar x)^2}{n+m-2}$$

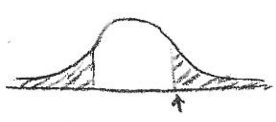
dvs. $T \sim t(n+m-2)$ under H_0, idet

$$\bar Z - \bar X \sim N\left(0, \left(\tfrac{1}{n} + \tfrac{1}{m}\right)\sigma^2\right) \text{ under } H_0$$

og
$$\left\{\begin{array}{l} \left.\begin{array}{l} \dfrac{\sum_i (z_i - \bar z)^2}{\sigma^2} \sim \chi^2(n-1) \\ \dfrac{\sum_j (X_j - \bar x)^2}{\sigma^2} \sim \chi^2(m-1) \end{array}\right\} \Rightarrow \\ \qquad \dfrac{(n+m-2)S^2}{\sigma^2} \sim \chi^2(n+m-2) \end{array}\right.$$

krit. omr. $(-\infty < t < -t_{1-\frac{\alpha}{2}}) \cup (t_{1-\frac{\alpha}{2}} < t < \infty)$

p-værdi $= 2P(T > |t_{obs}|)$



$|t_{obs}|$

Når der er forskellig varians i de to obs. rækker, foreligger Behrens-Fisher problemet, som ikke har nogen eksakt løsning, idet fordelingen under H_0 ikke kan bestemmes eksakt.

Der kan udledes et kvotienttest for ens varians i de to obs. rækker (et F-test, jf. opg. 4.8)

Teststørrelserne for test af ens variansen og for ens middelværdier er uafhængige. Udføres de to test på hhv. niveau α_1 og α_2, bliver sands. for accept af begge $(1-\alpha_1)(1-\alpha_2)$ $\approx 1-(\alpha_1+\alpha_2)$, dvs. samlet på niveau $\approx \alpha_1+\alpha_2$. Ønskes samlet et niveau på α, vælges normalt $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$.

Ved flere test på samme datagrundlag er der normalt ikke kontrol over det samlede niveau, da teststørrelserne ofte vil være afhængige.

t-test for parrede data

Typisk anvendelse når der måles / iagttages 'før' og 'efter' en begivenhed, fx en medicinsk behandling.

Antag
$$Z_i = \mu + W_i + E_{1i} \qquad W_i \sim N(0, \sigma_w^2)$$
$$X_i = \eta + W_i + E_{2i}$$

$$\Rightarrow Y_i = Z_i - X_i = \mu - \eta + E_i$$

dvs. ved at betragte differenser formindskes variansen i datamaterialet betragteligt.

Benyt sædv. t-test på y-data.

En-sidig variansanalyse

m uafh. obs. rækker ($m=2$ er behandlet)

$$Y_{ij} \sim N(\mu_i, \sigma^2) \qquad \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix} \qquad \text{uafh.}$$

H_0: $\mu_1 = \mu_2 = \dots = \mu_m \; (= \mu)$
H_1: mindst ét μ_i afvigende

Idé: Serie af t-test for ens middelværdi. Problemer med bestemmelse af et samlet signifikansniveau fører til afvisning af ideen.

I stedet udvikles et kvotienttest:

MLE: $\hat{\mu}_i = \bar{y}_{i\cdot}$, $\hat{\sigma}^2 = \frac{1}{mn} \sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot})^2$

under H_0:
$$\hat{\mu}_0 = \bar{y}_{\cdot\cdot}, \quad \hat{\sigma}_0^2 = \frac{1}{mn} \sum_i \sum_j (y_{ij} - \bar{y}_{\cdot\cdot})^2$$

$\longrightarrow$

$$\lambda(y) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-\frac{nm}{2}} = \left(\frac{\sum_i\sum_j (y_{ij}-\bar{y}_{..})^2}{\sum_i\sum_j (y_{ij}-\bar{y}_{i.})^2}\right)^{-\frac{nm}{2}}$$

$$= \left(\frac{\sum_i\sum_j \left((y_{ij}-\bar{y}_{i.}) + (\bar{y}_{i.}-\bar{y}_{..})\right)^2}{\sum_i\sum_j (y_{ij}-\bar{y}_{i.})^2}\right)^{-\frac{nm}{2}}$$

$$= \left(1 + \frac{n\sum_i (\bar{y}_{i.}-\bar{y}_{..})^2}{\sum_i\sum_j (y_{ij}-\bar{y}_{i.})^2}\right)^{-\frac{nm}{2}} = \left(1+\frac{d_0}{d}\right)^{-\frac{nm}{2}}$$

bemærk, at

$$\frac{D_0}{\sigma^2} = \frac{\sum_i (\bar{Y}_{i.}-\bar{Y}_{..})^2}{\frac{\sigma^2}{n}} \sim \chi^2(m-1) \quad \text{under } H_0,$$

idet $\bar{Y}_{i.} \sim N(\mu, \frac{\sigma^2}{n})$ under H_0

desuden

$$\frac{D}{\sigma^2} \sim \chi^2(m(n-1))$$

D_0 og D er uafh., da $D_0 = f(\bar{y}_{i.})$
og $D = g(\hat{\sigma}_i^2)$

sæt $$F = \frac{\frac{\frac{1}{\sigma^2}D_0}{m-1}}{\frac{\frac{1}{\sigma^2}D}{m(n-1)}} \sim F(m-1, m(n-1))$$
under H_0

krit. omr.: $f_{1-\alpha} < f < \infty$

Test af σ^2 i $N(\mu, \sigma^2)$

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 \neq \sigma_0^2$ (to-sidet test)

$\longrightarrow$

MLE: $\hat{\mu} = \bar{y}$, $\hat{\sigma}^2 = \frac{1}{n} \sum_i (y_i - \bar{y})^2$

Under H_0:

μ_0 er eneste parameter

$\hat{\mu}_0 = \bar{y}$ med de sædvanlige udregninger,

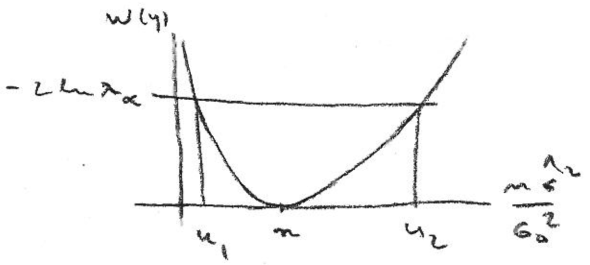
men $L(\hat{\mu}_0)$ bliver anderledes end det sædvanlige udtryk:

$$L(\hat{\mu}_0) = (2\pi \sigma_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2} n \hat{\sigma}^2\right)$$
$$= \left(2\pi \, e^{\frac{\hat{\sigma}^2}{\sigma_0^2}} \sigma_0^2\right)^{-\frac{n}{2}}$$

$$\lambda(y) = \frac{\left(2\pi e^{\frac{\hat{\sigma}^2}{\sigma_0^2}} \sigma_0^2\right)^{-\frac{n}{2}}}{\left(2\pi e \hat{\sigma}^2\right)^{-\frac{n}{2}}} = \left(\frac{\sigma_0^2}{\hat{\sigma}^2} e^{\frac{\hat{\sigma}^2}{\sigma_0^2} - 1}\right)^{-\frac{n}{2}}$$

$$W(y) = -2 \ln \lambda(y) = n \left(\ln \frac{\sigma_0^2}{\hat{\sigma}^2} + \frac{\hat{\sigma}^2}{\sigma_0^2} - 1 \right)$$
$$= -n \ln \frac{n\hat{\sigma}^2}{\sigma_0^2} + \frac{n\hat{\sigma}^2}{\sigma_0^2} + n \ln n - n$$

$W(y)$ afhænger kun af y gennem $\hat{\sigma}^2 \propto \frac{n\hat{\sigma}^2}{\sigma_0^2}$



$$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{(kendt)}$$
$$\Downarrow$$
$$U = \frac{n\hat{\sigma}^2}{\sigma_0^2} \sim \frac{\sigma^2}{\sigma_0^2} \chi^2(n-1)$$

Bestemmelse af u_1 og u_2:

$W(u_1) = W(u_2) \wedge P(u_1 < U < u_2) = 1 - \alpha$,

besværligt!

Traditionelt vælges c_1 og c_2, så

$P(U < c_1) = P(U > c_2) = \frac{\alpha}{2}$, som er en

god approksimation for store n

Test i binomialfordelingen

$Y \sim b(n, \theta)$

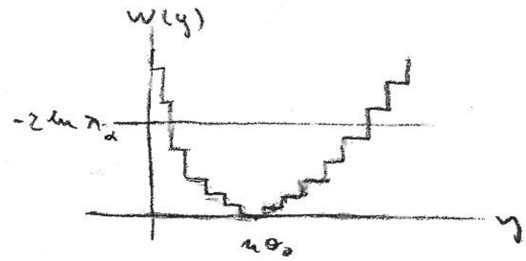
$H_0 : \theta = \theta_0$

$H_1 : \theta \neq \theta_0$ (to sidet test)

Kvotientteststørrelsen

$$\lambda(y) = \frac{\theta_0^y (1-\theta_0)^{n-y}}{\hat{\theta}^y (1-\hat{\theta})^{n-y}}$$, bemærk $\hat{\theta} = \frac{y}{n} \Rightarrow y = n\hat{\theta}$

$$= \frac{\theta_0^y (1-\theta_0)^{n-y}}{(\frac{y}{n})^y (1-\frac{y}{n})^{n-y}}$$



$W(y) = -2\ln\lambda(y)$

$\frac{dW(y)}{dy} = 0$ for $y = n\theta_0$

Krit. omr. $[0; y_1] \cup [y_2; n]$,

hvor $P(Y \leq y_1; \theta_0) + P(Y \geq y_2; \theta_0) \simeq \alpha$

Tal eks., se bog s. 134-135

Test i multinomialfordelingen

$(N_1, \dots, N_r) \sim m(n; \pi_1, \dots, \pi_r)$; $\sum_j \pi_j < 1$

$H_0 : \pi_j = p_j$, $j = 1, \dots, r$ | $\sum_j := \sum_{j=1}^{r}$ i udregningerne

$H_1 : \pi_j \neq p_j$ for mindst ét j

Sæt $p_0 = 1 - \sum_j p_j$, $\pi_0 = 1 - \sum_j \pi_j$, $n_0 = n - \sum_j n_j > 0$

$$\ell(\pi) = \sum_j n_j \ln \pi_j + n_0 \ln(1 - \sum_j \pi_j)$$

$$\frac{\partial \ell(\pi)}{\partial \pi_j} = \frac{n_j}{\pi_j} + \frac{n_0}{\pi_0}(-1) = 0 \Rightarrow \frac{n_j}{\pi_j} \text{ konst. } \left(= \frac{n_0}{\pi_0}\right)$$

$\longrightarrow$

$$n_j = k\pi_j \Rightarrow n = k \cdot 1 \quad \text{(summation fra 0 til } r\text{)}$$

dvs. $k = n$

altså $\hat{\pi}_j = \frac{n_j}{n}$

$$W = 2\left(l(\hat{\pi}) - l(p)\right) = 2\sum_{j=0}^{r} n_j \ln \frac{\hat{\pi}_j}{p_j} = G^2$$

$$\frac{\partial^2 l}{\partial \pi_i \partial \pi_j} = \delta_{ij}\left(-\frac{n_j}{\pi_j^2}\right) + \frac{n_0}{\pi_0^2}(-1)$$

$$E\left[-\frac{\partial^2 l}{\partial \pi_i \partial \pi_j}\right] = \delta_{ij}\frac{n\pi_j}{\pi_j^2} + \frac{n\pi_0}{\pi_0^2} = \delta_{ij}\frac{n}{\pi_j} + \frac{n}{\pi_0}$$

$$I(\pi) = \frac{n}{\pi_0}\left(D + 1_r 1_r^T\right), \quad D = \begin{bmatrix} \frac{\pi_0}{\pi_1} & & \\ & \ddots & \\ & & \frac{\pi_0}{\pi_r}\end{bmatrix}$$

$$(I(\pi))^{-1} = \frac{\pi_0}{n}\left(D^{-1} - \frac{1}{1 + 1_r^T D^{-1} 1_r} D^{-1} 1_r 1_r^T D^{-1}\right) \; *$$

$$\left((I(\pi))^{-1}\right)_{ij} = \frac{\pi_0}{n}\left(\delta_{ij}\frac{\pi_j}{\pi_0} - \frac{1}{1 + \sum_j \frac{\pi_j}{\pi_0}}\frac{\pi_i}{\pi_0}\frac{\pi_j}{\pi_0}\right)$$

$$= \frac{1}{n}\left(\delta_{ij}\pi_j - \frac{1}{1 + \frac{1-\pi_0}{\pi_0}}\frac{\pi_i \pi_j}{\pi_0}\right)$$

$$= \frac{1}{n}\left(\delta_{ij}\pi_j - \pi_i \pi_j\right)$$

$$\left((I(p))^{-1}\right)_{ij} = \frac{1}{n}\left(\delta_{ij} p_j - p_i p_j\right)$$

$$\left.\frac{\partial l(\pi)}{\partial \pi_j}\right|_{\pi = p} = \frac{n_j}{p_j} - \frac{n_0}{p_0}$$

$$W_u = \left[\left.\frac{dl}{d\pi}\right|_{\pi=p}\right]^T (I(p))^{-1}\left[\left.\frac{dl}{d\pi}\right|_{\pi=p}\right] \rightarrow$$

$$* \quad 1_r^T D^{-1} 1_r = [1 \dots 1]\begin{bmatrix}\frac{\pi_1}{\pi_0} & & \\ & \ddots & \\ & & \frac{\pi_r}{\pi_0}\end{bmatrix}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} = \left[\frac{\pi_1}{\pi_0} \dots \frac{\pi_r}{\pi_0}\right]\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = \sum_j \frac{\pi_j}{\pi_0}$$

Stat 23.3.2010

$$W_u = \frac{1}{n}\sum_i\sum_j\left(\frac{n_j}{p_j}-\frac{n_0}{p_0}\right)\left(\delta_{ij}p_j-p_ip_j\right)\left(\frac{n_i}{p_i}-\frac{n_0}{p_0}\right)$$

$$= \frac{1}{n}\left(\sum_i\left(\frac{n_i}{p_i}-\frac{n_0}{p_0}\right)^2 p_i\right.$$

$$\left. -\sum_i\sum_j \frac{n_ip_0-n_0p_i}{p_0p_i}\,p_ip_j\,\frac{n_jp_0-n_0p_j}{p_0p_j}\right)$$

$$= \frac{1}{n}\left(\sum_i\left(\frac{n_i^2}{p_i}-2\frac{n_in_0}{p_0}+\frac{n_0^2p_i}{p_0^2}\right)\right.$$

$$\left.-\sum_i\sum_j\frac{(n_ip_0-n_0p_i)(n_jp_0-n_0p_j)}{p_0^2}\right)$$

$$= \frac{1}{n}\left(\sum_i\frac{n_i^2}{p_i}-\frac{2(nn_0p_0-n_0^2p_0)-(n_0^2-n_0^2p_0)}{p_0^2}\right.$$

$$\left.-\frac{(np_0-n_0p_0-(n_0-n_0p_0))(np_0-n_0p_0-(n_0-n_0p_0))}{p_0^2}\right)$$

$$= \frac{1}{n}\left(\sum_i\frac{n_i^2}{p_i}-\frac{2nn_0p_0-n_0^2p_0-n_0^2+n^2p_0^2-2np_0n_0+n_0^2}{p_0^2}\right)$$

$$= \frac{1}{n}\left(\sum_i\frac{n_i^2}{p_i}-\left(n^2-\frac{n_0^2}{p_0}\right)\right)$$

$$= \frac{1}{n}\left(\sum_{i=0}^r\frac{n_i^2}{p_i}-n^2\right)$$

$$= \sum_{i=0}^r\frac{n_i^2}{np_i}-n$$

$$= \sum_{i=0}^r\frac{(n_i-np_i)^2}{np_i} = X^2 \quad \text{(Karl Pearson 1900)}$$

$$\left(=\sum_{i=0}^r\frac{n_i^2-2n_inp_i+(np_i)^2}{np_i}=\sum_{i=0}^r\frac{n_i^2}{np_i}-2n+n\right)$$

X^2 konv. hurtigere mod $\chi^2(r)$ end G^2

krav ved approksimation: alle $np_i > 2$ (5)

Bemærk $X^2 = \sum_i \frac{(\text{obs.}-\text{forv.})^2}{\text{forv.}}$

eks.

H_0: (U,V) ligefordelt på Q

H_1: (U,V) ikke ligefordelt på Q

Ikke-parametrisk test !

Approksimation :

Indlæg $m \times m$ gitter og notér antal obs. i hvert delkvadrat.

Benyt multinomialford. med $r = m^2 - 1$.

Fejlkilder ?.

Kontingenstabel

n obs. fordelt på $(r+1) \times (c+1)$ celler

test for homogenitet i søjler

H_0: $\pi_{ij} = \pi_{i0}$ for alle i,j

H_1: ikke alle lighedstegn holder

MLE : $\hat{\pi}_{ij} = \dfrac{n_{ij}}{n_{\cdot j}}$

under H_0 : $\hat{\pi}_{i0} = \dfrac{n_{i\cdot}}{n}$, $EN_{ij} = n_{\cdot j}\,\hat{\pi}_{i0} = \dfrac{n_{i\cdot}\,n_{\cdot j}}{n}$

$$G^2 = W = 2\left(\sum_{i=0}^{r} \sum_{j=0}^{c} n_{ij} \ln \frac{n_{ij}}{n_{\cdot j}} - \sum_{i=0}^{r} n_{i\cdot} \ln \frac{n_{i\cdot}}{n} \right)$$

$$X^2 = \sum_{i=0}^{r} \sum_{j=0}^{c} \frac{\left(n_{ij} - \frac{n_{i\cdot}\,n_{\cdot j}}{n}\right)^2}{\frac{n_{i\cdot}\,n_{\cdot j}}{n}} = \sum_i \sum_j \frac{(\text{obs.} - \text{forv.})^2}{\text{forv}}$$

test for homogenitet i rækker
test for uafhængighed mellem inddelingskriterier *
} de samme test-størrelser benyttes som ovenfor

$*\ EN_{ij} = n \dfrac{n_{i\cdot}}{n} \dfrac{n_{\cdot j}}{n} = \dfrac{n_{i\cdot}\,n_{\cdot j}}{n}$