6.7

6 a

$$\left(1-\frac{d}{dx}\right)^{(2)} \sin x = \left(1 - 2\frac{d}{dx} + \frac{d^2}{dx^2}\right) \sin x$$

$$= \sin x - 2\cos x - \sin x$$

$$= -2\cos x$$

b

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(3)} x y^3$$

$$= \left(h^3\frac{\partial^3}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2\partial y} + 3hk^2\frac{\partial^3}{\partial x\,\partial y^2} + k^3\frac{\partial^3}{\partial y^3}\right) x y^3$$

$$= 0 + 0 + 3\cdot 6\, h k^2 y + 6k^3$$

$$= 6k^2(kx + 3hy)$$

10 a

$$f(x,y) = x^3 - 4x^2 - xy - y^2$$

$$\frac{\partial f}{\partial x} = 3x^2 - 8x - y = 0$$

$$\frac{\partial f}{\partial y} = -x - 2y = 0 \Rightarrow x = -2y$$

$$\Rightarrow 12y^2 + 15y = 0$$

$$\Leftrightarrow y(4y+5) = 0$$

$$\Leftrightarrow y = 0 \lor y = -\frac{5}{4}$$

$$\Downarrow \qquad \Downarrow$$

$$x = 0 \qquad x = \frac{5}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = 6x - 8$$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -2$$

krit. pkt:er	A	B	C	D	art	$f(\cdot,\cdot)$
$(0,0)$	-8	-1	-2	15	l. maks.	0
$(\frac{5}{2}, -\frac{5}{4})$	7	-1	-2	-15	sadelpkt	$-\frac{125}{16}$

b

$$f(x,y) = (y - x^2)(2 - x - y)$$

$$\frac{\partial f}{\partial x} = -2x(2-x-y) + (y-x^2)(-1) = 0$$

$$\frac{\partial f}{\partial y} = 1(2-x-y) + (y-x^2)(-1) = 0$$

$$\Rightarrow -2x(2-x-y) = 2-x-y$$

$$\Rightarrow \underset{(1)}{2-x-y=0} \lor \underset{(2)}{x = -\tfrac{1}{2}}$$

(1) $\Rightarrow$ $y = x^2$ $\Rightarrow$ $x^2 + x - 2 = 0$ $\Rightarrow$ $x = 1 \lor x = -2$

$$\Downarrow \qquad \Downarrow$$

$$y = 1 \qquad y = 4$$

(2) $\Rightarrow$ $2 + \frac{1}{2} - y - y + \frac{1}{4} = 0$ $\Rightarrow$ $y = \frac{11}{8}$

$$\frac{\partial^2 f}{\partial x^2} = -2(2-x-y) - 2x(-1) + 2x = 6x + 2y - 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x - 1$$

$$\frac{\partial^2 f}{\partial y^2} = -1 - 1 = -2$$

fortsættes

6.8

10 b fortsat

krit. pkt.'er	A	B	C	D	art	$f(\cdot,\cdot)$
$(-2,4)$	-8	-5	-2	-9	sadelpkt	0
$(-\frac{1}{2}, \frac{11}{8})$	$-\frac{17}{4}$	-2	-2	$\frac{1}{4}$	l. maks.	$\frac{81}{64}$
$(1,1)$	4	1	-2	-9	sadelpkt.	0

6.9

8 $V = \frac{1}{3}\pi R^2 h$, bibet : $A = \pi R s$ konst.

$\Leftrightarrow \pi R\sqrt{R^2+h^2} - A = 0$

$$f(R,h,\lambda) = \frac{1}{3}\pi R^2 h - \lambda\left(\pi R\sqrt{R^2+h^2} - A\right)$$

$$\frac{\partial f}{\partial R} = \frac{2}{3}\pi R h - \lambda\pi\left(\sqrt{R^2+h^2} + R\frac{2R}{2\sqrt{R^2+h^2}}\right) = 0 \qquad (1)$$

$$\frac{\partial f}{\partial h} = \frac{1}{3}\pi R^2 - \lambda\pi R\frac{2h}{2\sqrt{R^2+h^2}} = 0 \qquad (2)$$

$$\left.\begin{array}{l}(1) \Rightarrow \frac{2}{3}Rh - \lambda\frac{2R^2+h^2}{\sqrt{R^2+h^2}} = 0 \\ (2) \Rightarrow \frac{2}{3}Rh - \lambda\frac{2h^2}{\sqrt{R^2+h^2}} = 0\end{array}\right\} \Rightarrow h^2 = 2R^2 \Rightarrow h = \sqrt{2}R$$

inds. i bibet.

$$A = \pi R\sqrt{1+2}R = \pi\sqrt{3}R^2 \Rightarrow R = \frac{\sqrt{A}}{\sqrt[4]{3}\sqrt{\pi}}$$

$$V = \frac{1}{3}\pi\frac{A}{\sqrt{3}\pi}\sqrt{2}\frac{\sqrt{A}}{\sqrt[4]{3}\sqrt{\pi}} = \frac{\sqrt{2}A\sqrt{A}}{3\sqrt{3}\sqrt[4]{3}\sqrt{\pi}} = \frac{2^{\frac{1}{2}}A^{\frac{3}{2}}}{3^{\frac{7}{4}}\pi^{\frac{1}{2}}}$$

18 $f(x_1, x_2, \dots, x_n) = x_1 x_2 \dots x_n$

bibet : $x_1 + x_2 + \dots + x_n = a \Leftrightarrow x_1 + x_2 + \dots + x_n - a = 0$

$$g(x_1, \dots, x_n, \lambda) = x_1 x_2 \dots x_n - \lambda(x_1 + x_2 + \dots + x_n - a)$$

$$\frac{\partial g}{\partial x_i} = x_1 \dots x_{i-1} x_{i+1} \dots x_n - \lambda = 0 \qquad (n \text{ ligninger})$$

$\Rightarrow x_1 \dots x_n - \lambda x_i = 0 \quad , \quad i = 1, \dots, n \qquad *$

$\Rightarrow \sum_{i=1}^{n} x_1 \dots x_n - \lambda\sum_{i=1}^{n} x_i = 0$

$\Rightarrow n\, x_1 \dots x_n - \lambda a = 0$

$\Rightarrow \lambda = \frac{n}{a} x_1 \dots x_n$ indsættes i *

$\Rightarrow x_i = \frac{a}{n} \quad , \quad i = 1, \dots, n$

$\Rightarrow x_1 x_2 \dots x_n = \left(\frac{a}{n}\right)^n$

7.1

6 $$\operatorname{div}\frac{\bar r}{r^3} = \operatorname{div}\left(\frac{x}{r^3}\bar\imath + \frac{y}{r^3}\bar\jmath + \frac{z}{r^3}\bar k\right)$$
$$= \frac{1}{r^3} - \frac{3x}{r^4}\frac{2x}{2r} + \frac{1}{r^3} - \frac{3y}{r^4}\frac{2y}{2r} + \frac{1}{r^3} - \frac{3z}{r^4}\frac{2z}{2r}$$
$$= \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

7 $$\nabla\frac{1}{r} = -\frac{1}{r^2}\,\frac{2x\bar\imath + 2y\bar\jmath + 2z\bar k}{2r} = -\frac{\bar r}{r^3}$$

8 $$\nabla^2\frac{1}{r} = \operatorname{div}\left(\nabla\frac{1}{r}\right) = \operatorname{div}\left(-\frac{\bar r}{r^3}\right) = 0$$, jf. 7.1 7 og 7.1 6

9 $$\nabla\times(f(r)\bar r) = \begin{vmatrix} \bar\imath & \bar\jmath & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \left(z f'(r)\frac{2y}{2r} - y f'(r)\frac{2z}{2r}\right)\bar\imath$$
$$+ \left(x f'(r)\frac{2z}{2r} - z f'(r)\frac{2x}{2r}\right)\bar\jmath$$
$$+ \left(y f'(r)\frac{2x}{2r} - x f'(r)\frac{2y}{2r}\right)\bar k$$
$$= \bar 0$$

14 a $$\operatorname{div}(\varphi\bar v) = \frac{\partial}{\partial x}(\varphi v_1) + \frac{\partial}{\partial y}(\varphi v_2) + \frac{\partial}{\partial z}(\varphi v_3)$$
$$= \frac{\partial\varphi}{\partial x}v_1 + \varphi\frac{\partial v_1}{\partial x} + \frac{\partial\varphi}{\partial y}v_2 + \varphi\frac{\partial v_2}{\partial y} + \frac{\partial\varphi}{\partial z}v_3 + \varphi\frac{\partial v_3}{\partial z}$$
$$= \varphi\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)$$
$$+ \left(\frac{\partial\varphi}{\partial x}\bar\imath + \frac{\partial\varphi}{\partial y}\bar\jmath + \frac{\partial\varphi}{\partial z}\bar k\right)\cdot(v_1\bar\imath + v_2\bar\jmath + v_3\bar k)$$
$$= \varphi\operatorname{div}\bar v + \nabla\varphi\cdot\bar v$$

b $\varphi = xy$, $\bar v = y^2\bar\imath + xz\bar k$

$$\operatorname{div}(\varphi\bar v) = xy(0+0+x) + (y\bar\imath + x\bar\jmath)\cdot(y^2\bar\imath + xz\bar k)$$
$$= x^2y + y^3$$

c $$\varphi\bar v = xy(y^2\bar\imath + xz\bar k) = xy^3\bar\imath + x^2yz\bar k$$

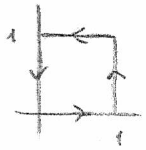
$$\operatorname{div}(\varphi\bar v) = y^3 + x^2y$$

7.2

5 $\bar{A} = x^2y\,\bar{i} + y^2z\,\bar{j} + z^2x\,\bar{k}$ $\qquad C: \bar{r}(u) = u\bar{i} + u\bar{j} + 3u\bar{k}$, $0 \le u \le 1$

$$\int_C A \cdot d\bar{r} = \int_0^1 (u^3 \cdot 1 + 3u^3 \cdot 1 + 9u^3 \cdot 3)\,du = \int_0^1 31u^3\,du$$
$$= \left[31\frac{u^4}{4}\right]_0^1 = \frac{31}{4}$$

7 $\bar{A} = xy\,\bar{i} + y^2\,\bar{j}$, ikke konservativt, da $\frac{\partial y^2}{\partial x} \neq \frac{\partial xy}{\partial y}$



$$\oint \bar{A} \cdot d\bar{r} = \int_0^1 0 \cdot 1\,dx + \int_0^1 y^2 \cdot 1\,dy + \int_1^0 x \cdot 1\,dx + \int_1^0 y^2 \cdot 1\,dy$$
$$= \left[\frac{x^2}{2}\right]_1^0 = -\frac{1}{2}$$

8 $\bar{A} = (3x^2 + 2y^2)\,\bar{i} + (4xy - 6y^2)\,\bar{j}$

$$\frac{\partial(4xy - 6y^2)}{\partial x} = \frac{\partial(3x^2 + 2y^2)}{\partial y} \quad (= 4y), \text{ dvs. } \bar{A} \text{ er konservativt}$$

$$\left.\begin{aligned} \int (3x^2 + 2y^2)\,dx &= x^3 + 2y^2x + g(y) \\ \int (4xy - 6y^2)\,dy &= 2xy^2 + 2y^3 + h(x) \end{aligned}\right\} \Rightarrow \varphi(x,y) = x^3 + 2xy^2 + 2y^3 + c$$

7.3

5 $z = 2x + 3y$ (plan) inden for $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (elliptisk cylinder)

$\bar{N} = -2\bar{i} - 3\bar{j} + \bar{k}$, $|\bar{N}| = \sqrt{4+9+1} = \sqrt{14}$

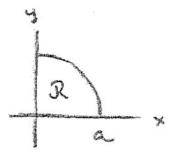
$Ar = \int_S dS = \int_R |\bar{N}|\, dA_{xy} = \sqrt{14}\, A_{xy}$

$A_{xy} = \pi \cdot 2 \cdot 3 = 6\pi$ (areal af ellipse)

$Ar = 6\pi \sqrt{14}$

10 $f(x,y,z) = x$, $S: x^2 + y^2 + z^2 = a^2$ i første oktant

$\Leftrightarrow \quad z^2 = a^2 - x^2 - y^2$



$\bar{N} = -\frac{1}{2\sqrt{a^2-x^2-y^2}}(-2x\bar{i} - 2y\bar{j}) + \bar{k}$

$|\bar{N}| = \sqrt{\frac{x^2+y^2+(a^2-x^2-y^2)}{a^2-x^2-y^2}} = \frac{a}{\sqrt{a^2-r^2}}$

$$\int_S f\, dS = \int_R x\,|\bar{N}|\, dA_{xy} = \int_0^{\frac{\pi}{2}} \int_0^a r\cos\theta \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

$$= [\sin\theta]_0^{\frac{\pi}{2}}\, a \int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\, dr$$

$$= 1 \cdot a \left[-\frac{r}{2}\sqrt{a^2-r^2} + \frac{a^2}{2} \arcsin\frac{r}{a} \right]_0^a$$

$$= a\left(0 + \frac{a^2}{2}\frac{\pi}{2}\right) = \frac{\pi}{4} a^3$$

7.4

9 $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$, $\mathcal{S}: x^2+y^2 = 4$, $-1 \le z \le 1$
+ endeflader

$$\int_{\mathcal{S}} \bar{F}\cdot\bar{n}\, dS = \int_{\mathcal{V}} \operatorname{div}\bar{F}\, dV = \int_{\mathcal{V}} (3x^2+3y^2+3z^2)\, dV$$
$$= 2\cdot 3 \int_0^{2\pi}\int_0^2\int_0^1 (r^2+z^2)\, r\, dz\, dr\, d\theta$$
$$= 6\cdot 2\pi \int_0^2 \left[r^3 z + r\frac{z^3}{3} \right]_0^1 dr$$
$$= 12\pi \int_0^2 \left(r^3 + \frac{r}{3}\right) dr = 12\pi \left[\frac{r^4}{4} + \frac{r^2}{6} \right]_0^2$$
$$= 56\pi$$

11
$$\int_{\mathcal{S}} f\frac{\partial g}{\partial n}\, dS = \int_{\mathcal{S}} f\,\nabla g\cdot\bar{n}\, dS = \int_{\mathcal{V}} \operatorname{div}(f\nabla g)\, dV$$
$$= \int_{\mathcal{V}} (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV$$, jf. 7.1 14a

12
$$\int_{\mathcal{S}} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS = \int_{\mathcal{V}} \operatorname{div}(f\nabla g - g\nabla f)\, dV$$
$$= \int_{\mathcal{V}} (f\nabla^2 g + \nabla f\cdot\nabla g - g\nabla^2 f - \nabla g\cdot\nabla f)\, dV$$
$$= \int_{\mathcal{V}} (f\nabla^2 g - g\nabla^2 f)\, dV$$

7.5

5 $\bar{v} = (x+y-z)\,\bar{i} + xy\,\bar{j} + xz\,\bar{k}$

$\mathcal{S}: \quad z = 16 - (x^2+y^2), \quad z \geq 0$

$C: \quad x^2+y^2 = 16, \quad z = 0$

$\bar{r}(\theta) = 4\cos\theta\,\bar{i} + 4\sin\theta\,\bar{j}$

$$\int_{\mathcal{S}} \operatorname{curl}\bar{v}\cdot\bar{n}\,dS = \oint_C \bar{v}\cdot d\bar{r}$$

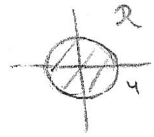
$$= \int_0^{2\pi} \left((4\cos\theta + 4\sin\theta - 2)(-4\sin\theta) + 4\cos\theta\cdot 4\sin\theta\cdot 4\cos\theta + 0 \right) d\theta$$

$$= \int_0^{2\pi} \left(-16\cos\theta\sin\theta - 16\sin^2\theta + 8\sin\theta + 64\cos^2\theta\sin\theta \right) d\theta$$

$$= 0 - 16\,\tfrac{1}{2}\cdot 2\pi + 0 + 0$$

$$= -16\pi$$

Kontroludregning:



$$\int_{\mathcal{S}} \operatorname{curl}\bar{v}\cdot\bar{n}\,dS = \int_{\mathcal{R}} \operatorname{curl}\bar{v}\cdot\bar{N}\,dA_{xy}$$

$$= \int_{\mathcal{R}} \left(-z\,\bar{j} + (y-1)\,\bar{k}\right)\cdot\left(2x\,\bar{i} + 2y\,\bar{j} + \bar{k}\right) dA_{xy}$$

$$= \int_0^{2\pi}\int_0^4 (0 + r\sin\theta - 1)\, r\,dr\,d\theta$$

$$= 0 - 2\pi\left[\frac{r^2}{2}\right]_0^4$$

$$= -16\pi$$

22 $\bar{F} = \operatorname{grad} f$, $\mathcal{S}$ med tilhørende randkurve C vælges arbitrært

$$\int_{\mathcal{S}} \operatorname{curl}\bar{F}\cdot\bar{n}\,dS = \oint_C \bar{F}\cdot d\bar{r} = \oint_C \operatorname{grad} f\cdot d\bar{r} = 0$$

$$\Rightarrow \operatorname{curl}\bar{F} = \bar{0} \iff \operatorname{curl}\operatorname{grad} f = \bar{0}$$

23 $$\oint_C P\,dx + Q\,dy = \oint_C (P\bar{i} + Q\bar{j})\cdot d\bar{r}$$

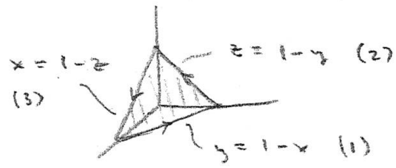
$$= \int_{\mathcal{R}} \operatorname{curl}(P\bar{i} + Q\bar{j})\,dA_{xy}$$

$$= \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA_{xy}$$

7.5

6 $\bar{F} = (2y+z)\,\bar{i} + (x-z)\,\bar{j} + (y-x)\,\bar{k}$

$\mathcal{S}$: $x+y+z=1$ i første oktant



(1) $\bar{r}(x) = x\,\bar{i} + (1-x)\,\bar{j}$, $0 \le x \le 1$

(2) $\bar{r}(y) = y\,\bar{j} + (1-y)\,\bar{k}$, $0 \le y \le 1$

(3) $\bar{r}(z) = (1-z)\,\bar{i} + z\,\bar{k}$, $0 \le z \le 1$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_1^0 (2(1-x)\cdot 1 + x(-1)\,dx$$
$$+ \int_1^0 (-(1-y)\cdot 1 + y(-1))\,dy$$
$$+ \int_1^0 (z(-1) - (1-z)\cdot 1)\,dz$$
$$= \int_1^0 (2-3x)\,dx - \int_1^0 dy - \int_1^0 dz$$
$$= \left[2x - 3\frac{x^2}{2}\right]_1^0 + 1 + 1$$
$$= -2 + \frac{3}{2} + 2$$
$$= \frac{3}{2}$$

$$\int_{\mathcal{S}} \text{curl}\,\bar{F}\cdot\bar{n}\,dS = \int_{\mathcal{R}} \text{curl}\,\bar{F}\cdot\bar{N}\,dA_{xy}$$
$$= \int_{\mathcal{R}} (2\bar{i} + 2\bar{j} - \bar{k})\cdot(\bar{i}+\bar{j}+\bar{k})\,dA_{xy}$$
$$= \int_{\mathcal{R}} 3\,dA_{xy}$$
$$= 3\left(\tfrac{1}{2}\cdot 1\cdot 1\right)$$
$$= \frac{3}{2}$$

$(z = -x-y+1 \Rightarrow \bar{N} = -(-1)\bar{i} - (-1)\bar{j} + \bar{k} = \bar{i}+\bar{j}+\bar{k})$

8.1

7 $\quad r = a + b\cos\theta \quad , \quad a > b$

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} = \sqrt{(-b\sin\theta)^2 + (a+b\cos\theta)^2}$$

$$= \sqrt{b^2\sin^2\theta + a^2 + 2ab\cos\theta + b^2\cos^2\theta}$$

$$= \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

$$s = 2\int_0^{\pi} \sqrt{a^2+b^2+2ab\cos\theta}\; d\theta$$

$$= 2\int_0^{\pi} \sqrt{a^2+b^2+2ab - 2ab(1-\cos\theta)}\; d\theta$$

$$= 2\int_0^{\pi} \sqrt{(a+b)^2 - 4ab\sin^2\frac{\theta}{2}}\; d\theta \quad , \qquad u = \frac{\theta}{2}, \quad du = \frac{1}{2}\,d\theta$$

$$= 4(a+b)\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{4ab}{(a+b)^2}\sin^2 u}\; du$$

$$= 4(a+b)\, E\left[\frac{2\sqrt{ab}}{a+b}\right]$$

8.2

7 $f(r,\theta) = r^2 - a^2 \cos\theta$

$$\nabla f = \frac{\partial f}{\partial r}\,\bar{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\bar{e}_\theta$$

$$= 2r\,\bar{e}_r + \frac{1}{r}\,a^2 \sin\theta\,\bar{e}_\theta$$

8 $\bar{u} = r\cos\theta\,\bar{e}_r - r\sin\theta\,\bar{u}_\theta$

$$\operatorname{div}\bar{u} = \frac{1}{r}\left(\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$= \frac{1}{r}\,(2r\cos\theta - r\cos\theta)$$

$$= \cos\theta$$

8.3

11 $$\nabla \times \bar{u} = \left(\bar{e}_r \frac{\partial}{\partial r} + \frac{1}{r} \bar{e}_\theta \frac{\partial}{\partial \theta} + \bar{e}_z \frac{\partial}{\partial z} \right) \times \left(u_r \bar{e}_r + u_\theta \bar{e}_\theta + u_z \bar{e}_z \right)$$

$$= \bar{e}_r \times \left(\frac{\partial u_r}{\partial r} \bar{e}_r + \bar{0} + \frac{\partial u_\theta}{\partial r} \bar{e}_\theta + \bar{0} + \frac{\partial u_z}{\partial r} \bar{e}_z + \bar{0} \right)$$

$$+ \frac{1}{r} \bar{e}_\theta \times \left(\frac{\partial u_r}{\partial \theta} \bar{e}_r + u_r \bar{e}_\theta + \frac{\partial u_\theta}{\partial \theta} \bar{e}_\theta + u_\theta (-\bar{e}_r) + \frac{\partial u_z}{\partial \theta} \bar{e}_z + \bar{0} \right)$$

$$+ \bar{e}_z \times \left(\frac{\partial u_r}{\partial z} \bar{e}_r + \bar{0} + \frac{\partial u_\theta}{\partial z} \bar{e}_\theta + \bar{0} + \frac{\partial u_z}{\partial z} \bar{e}_z + \bar{0} \right)$$

13 $$= \bar{0} + \frac{\partial u_\theta}{\partial r} \bar{e}_r \times \bar{e}_\theta + \frac{\partial u_z}{\partial r} \bar{e}_r \times \bar{e}_z$$

$$+ \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} \bar{e}_\theta \times \bar{e}_r + \bar{0} - u_\theta \bar{e}_\theta \times \bar{e}_r + \frac{\partial u_z}{\partial \theta} \bar{e}_\theta \times \bar{e}_z \right)$$

$$+ \frac{\partial u_r}{\partial z} \bar{e}_z \times \bar{e}_r + \frac{\partial u_\theta}{\partial z} \bar{e}_z \times \bar{e}_\theta + \bar{0}$$

$$= \frac{\partial u_\theta}{\partial r} \bar{e}_z - \frac{\partial u_z}{\partial r} \bar{e}_\theta$$

$$+ \frac{1}{r} \left(- \frac{\partial u_\theta}{\partial \theta} \bar{e}_z + u_\theta \bar{e}_z + \frac{\partial u_z}{\partial \theta} \bar{e}_r \right)$$

$$+ \frac{\partial u_r}{\partial z} \bar{e}_\theta - \frac{\partial u_\theta}{\partial z} \bar{e}_r$$

$$= \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \bar{e}_r$$

$$+ \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \bar{e}_\theta$$

$$+ \left(\frac{1}{r} u_\theta + \frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \bar{e}_z$$

$$= \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \bar{e}_r$$

$$+ \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \bar{e}_\theta$$

$$+ \frac{1}{r} \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \bar{e}_z$$

17 $$= \frac{1}{r} \begin{vmatrix} \bar{e}_r & r\bar{e}_\theta & \bar{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix}$$

8 $\quad S$: enhedskuglen , $dS = \sin\theta \, d\theta \, d\varphi$

$$\int_S \sin^2\theta \cos^2\theta \, dS = \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \cos^2\theta \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi \int_0^{\pi} -(1-\cos^2\theta)\cos^2\theta \, d(\cos\theta)$$

$$= 2\pi \int_0^{\pi} (-\cos^2\theta + \cos^4\theta) \, d(\cos\theta)$$

$$= 2\pi \left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5} \right]_0^{\pi}$$

$$= 2\pi \left(\frac{2}{3} - \frac{2}{5} \right)$$

$$= \frac{8\pi}{15}$$

14 $\quad f(r,\theta,\varphi) = r^2 \sin^2\theta \sin 2\varphi$

$$\frac{\partial f}{\partial r} = 2r\sin^2\theta \sin 2\varphi \quad , \quad \frac{2}{r}\frac{\partial f}{\partial r} = 4\sin^2\theta \sin 2\varphi$$

$$\frac{\partial f}{\partial \theta} = 2r^2 \sin\theta\cos\theta \sin 2\varphi \quad , \quad \sin\theta \frac{\partial f}{\partial \theta} = 2r^2\sin^2\theta\cos\theta \sin 2\varphi$$

$$\frac{\partial f}{\partial \varphi} = 2r^2\sin^2\theta \cos 2\varphi$$

$$\frac{\partial^2 f}{\partial r^2} = 2\sin^2\theta \sin 2\varphi$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) = 4r^2\sin\theta\cos^2\theta \sin 2\varphi - 2r^2\sin^2\theta\sin\theta\sin 2\varphi$$

$$= 4r^2\sin\theta\cos^2\theta\sin 2\varphi - 2r^2\sin^3\theta \sin 2\varphi$$

$$\frac{\partial^2 f}{\partial\varphi^2} = -4r^2\sin^2\theta \sin 2\varphi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$= (2\sin^2\theta + 4\sin^2\theta + 4\cos^2\theta - 2\sin^2\theta - 4)\sin 2\varphi$$

$$= 0$$

8.5

8 $x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\,(r\cos^2\theta + r\sin^2\theta)$$

$$= r$$

9 $x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2\cos\theta\sin\theta\cos^2\varphi + r^2\cos\theta\sin\theta\sin^2\varphi)$$
$$+ r\sin\theta\,(r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi)$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta$$

15.1

11 $$f(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0 \\ 1 & \text{for } 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} 1 \, dx = 1$$

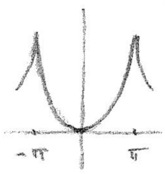
$$a_n = \frac{1}{\pi} \int_0^{\pi} 1 \cos nx \, dx = \frac{1}{n\pi} \left[\sin nx \right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} 1 \sin nx \, dx = \frac{1}{n\pi} \left[-\cos nx \right]_0^{\pi} = \begin{cases} \frac{1}{n\pi}(1+1) & n \text{ ulige} \\ \frac{1}{n\pi}(-1+1) & n \text{ lige} \end{cases}$$

$$= \frac{2}{(2n-1)\pi} \quad \text{med } n := 2n-1$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

12 $$f(x) = x^2 \,, \quad -\pi \le x \le \pi$$



$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \left[x^2 \frac{1}{n} \sin nx \right]_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} 2x \sin nx \, dx$$

$$= 0 - \frac{4}{n\pi} \left[x \left(-\frac{1}{n} \cos nx \right) \right]_0^{\pi} + \frac{4}{n^2\pi} \int_0^{\pi} 1 \cos nx \, dx$$

$$= \frac{4}{n^2\pi} \pi (-1)^n + \frac{4}{n^2\pi} \left[\sin nx \right]_0^{\pi} = \frac{(-1)^n 4}{n^2} + 0$$

$b_n = 0$, da $f(x)$ lige

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

13 $$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ \sin x & \text{for } 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin x \, dx = \frac{1}{\pi} \left[-\cos x \right]_0^{\pi} = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} \sin x \, d(\sin x) = \frac{1}{\pi} \left[\frac{\sin^2 x}{2} \right]_0^{\pi}$$

$$= 0$$

fortsættes

15.1

13 fortsat

$$a_n = \frac{1}{\pi}\int_0^\pi \sin x \cos nx\,dx = \frac{1}{\pi}\left[-\frac{\cos(1-n)x}{2(1-n)} - \frac{\cos(1+n)}{2(1+n)}\right]_0^\pi$$

$$= \begin{cases} 0 & n \text{ ulige} \\ \frac{1}{\pi}\left(\frac{1}{1-n}+\frac{1}{1+n}\right) = \frac{1}{\pi}\,\frac{2}{1-n^2} & n \text{ lige}\end{cases}$$

$$= \frac{2}{\pi}\,\frac{1}{1-(2n)^2} = -\frac{2}{\pi}\,\frac{1}{4n^2-1}\,, \quad n \ge 2$$

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x\,dx = \frac{1}{\pi}\int_0^\pi \sin^2 x\,dx = \frac{1}{\pi}\,\frac{1}{2}\pi = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin x \sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin(1-n)x}{2(1-n)} - \frac{\sin(1+n)x}{2(1+n)}\right]_0^\pi = 0$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx$$

16 $\quad f(x) = x^4\,, \quad 0 \le x \le 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^4\,dx = \frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{2\pi} = \frac{32\pi^4}{5}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^4 \cos nx\,dx = \frac{1}{\pi}\left(\left[x^4\,\frac{1}{n}\sin nx\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} 4x^3 \sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(0 - \frac{4}{n}\left[x^3\left(-\frac{1}{n}\cos nx\right)\right]_0^{2\pi} - \frac{4}{n^2}\int_0^{2\pi} 3x^2 \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{4}{n^2}\,8\pi^3 - \frac{12}{n^2}\left[x^2\,\frac{1}{n}\sin nx\right]_0^{2\pi} + \frac{12}{n^3}\int_0^{2\pi} 2x \sin nx\right)$$

$$= \frac{1}{\pi}\left(\frac{32\pi^3}{n^2} - 0 + \frac{24}{n^3}\left[x\left(-\frac{1}{n}\cos nx\right)\right]_0^{2\pi} + \frac{24}{n^4}\int_0^{2\pi} 1 \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{32\pi^3}{n^2} - \frac{24}{n^4}\,2\pi + \frac{24}{n^4}\left[\frac{1}{n}\sin nx\right]_0^{2\pi}\right) = \frac{32\pi^2}{n^2} - \frac{48}{n^4} + 0$$

$$= 16\left(\frac{2\pi^2}{n^2} - \frac{3}{n^4}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^4 \sin nx\,dx = \frac{1}{\pi}\left(\left[x^4\left(-\frac{1}{n}\cos nx\right)\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} 4x^3 \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\,16\pi^4 + \frac{4}{n}\left[x^3\,\frac{1}{n}\sin nx\right]_0^{2\pi} - \frac{4}{n^2}\int_0^{2\pi} 3x^2 \sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{16\pi^4}{n} + 0 - \frac{12}{n^2}\left[x^2\left(-\frac{1}{n}\cos nx\right)\right]_0^{2\pi} - \frac{12}{n^3}\int_0^{2\pi} 2x \cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{16\pi^4}{n} + \frac{12}{n^3}\,4\pi^2 - \frac{24}{n^3}\left[x\,\frac{1}{n}\sin nx\right]_0^{2\pi} + \frac{24}{n^4}\int_0^{2\pi} \sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{16\pi^4}{n} + \frac{48\pi^2}{n^3} - 0 + \frac{24}{n^4}\left[-\frac{1}{n}\cos nx\right]_0^{2\pi}\right)$$

$$= -\frac{16\pi^3}{n} + \frac{48\pi}{n^3} + 0 = 16\pi\left(\frac{3}{n^3} - \frac{\pi^2}{n}\right)$$

fortsættes

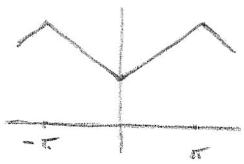
16 fortsat

$$f(x) = \frac{16\pi^4}{5} + 16\sum_{n=1}^{\infty}\left(\left(\frac{2\pi^2}{n^2} - \frac{3}{n^4}\right)\cos nx + \pi\left(\frac{3}{n^3} - \frac{\pi^2}{n}\right)\sin nx\right)$$

15.2

7 $f(x) = \pi + x$, $0 \le x \le \pi$

udvides til en lige fkt.



$$a_0 = \frac{2}{\pi}\int_0^\pi (\pi+x)\,dx = \frac{2}{\pi}\left[\pi x + \frac{x^2}{2}\right]_0^\pi = \frac{2}{\pi}\left(\pi^2+\frac{\pi^2}{2}\right) = 3\pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi+x)\cos nx\,dx = \frac{2}{\pi}\left(\left[(\pi+x)\frac{1}{n}\sin nx\right]_0^\pi - \frac{1}{n}\int_0^\pi 1 \sin nx\,dx\right)$$

$$= \frac{2}{\pi}\left(0 - \frac{1}{n}\left[-\frac{1}{n}\cos nx\right]_0^\pi\right) = \begin{cases} -\frac{2}{n\pi}\cdot\frac{2}{n} & n \text{ ulige} \\ 0 & n \text{ lige}\end{cases}$$

$$= -\frac{4}{(2n-1)^2\pi}$$

$b_n = 0$ da f lige

$$f(x) = \frac{3\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

15.3

2 $$f(x) = \ell^2 - x^2 = \frac{2\ell^2}{3} + \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{\ell}$$ jf. s. 715 – 716

$$\frac{1}{\ell}\int_{-\ell}^{\ell}(\ell^2-x^2)^2\,dx = \frac{2}{\ell}\left[\ell^4 x - 2\ell^2\frac{x^3}{3} + \frac{x^5}{5}\right]^{\ell} = 2\left(\ell^4 - \frac{2}{3}\ell^4 + \frac{1}{5}\ell^4\right) = \frac{16\ell^4}{15}$$

$$\frac{16\ell^4}{15} = \frac{1}{2}\left(\frac{4\ell^2}{3}\right)^2 + \left(\frac{4\ell^2}{\pi^2}(-1)^{n+1}\right)^2\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)^2$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \left(\frac{16\ell^4}{15} - \frac{8\ell^4}{9}\right)\frac{\pi^4}{16\ell^4} = \frac{\pi^4}{90}$$

11 $$f(x) = x^2 = \frac{4\ell^2}{3} + \frac{4\ell^2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}\cos\frac{n\pi x}{\ell} - \frac{1}{n}\sin\frac{n\pi x}{\ell}\right)$$ jf. s. 720

$\ell = \pi$: $$f(x) = \frac{4\pi^2}{3} + 4\pi\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}\cos nx - \frac{1}{n}\sin nx\right)$$

12 $$x^2 = \frac{4\ell^2}{\pi^2}\left(\frac{\pi^2}{12} - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{\ell}\right) = \frac{\ell^3}{3} - \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{\ell}$$

$$\frac{1}{3}x^3 = \frac{\ell^2 x}{3} - \frac{4\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[\frac{\ell}{n\pi}\sin\frac{n\pi\xi}{\ell}\right]_0^x$$

$$x^3 = \ell^2 x - \frac{12\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin\frac{n\pi x}{\ell} \qquad *$$

Fra eks. 3 s. 735 : $x = \frac{2\ell}{\pi}\sum\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{\ell}$ indsættes i *

$$x^3 = \frac{2\ell^3}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{\ell} - \frac{12\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin\frac{n\pi x}{\ell}$$

13 Fra * : $$\frac{1}{4}x^4 = \ell^2\frac{1}{2}x^2 - \frac{12\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\left[-\frac{\ell}{n\pi}\cos\frac{n\pi\xi}{\ell}\right]_0^x$$

$$x^4 = 2\ell^2x^2 - \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\left(1 - \cos\frac{n\pi x}{\ell}\right)$$

$x = \ell$: $$\ell^4 = 2\ell^4 + \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} + \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}(-1)^n \quad -\frac{1}{n^4}$$

$$= 2\ell^4 + \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} - \frac{48\ell^4}{\pi^4}\frac{\pi^4}{90}$$ jf. opg. 2

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{\pi^4}{48}\left(1 - \frac{24}{45}\right) = \frac{7\pi^4}{720}$$

15.4

16 $\quad y'' + 2y' + 3y = t \quad , \quad -\pi \le t \le \pi$

Fra eks. 2 s. 716 : $\quad t = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt$

Gæt :
$$\left.\begin{aligned} y_n &= c_n \cos nt + d_n \sin nt \\ y_n' &= -c_n n \sin nt + d_n n \cos nt \\ y_n'' &= -c_n n^2 \cos nt - d_n n^2 \sin nt \end{aligned}\right\} \text{ indsætter}$$

$$\begin{cases} (3-n^2)\,c_n + 2n\,d_n = 0 \\ -2n\,c_n + (3-n^2)\,d_n = \dfrac{2(-1)^{n+1}}{n} \end{cases}$$

$$c_n = \frac{\begin{vmatrix} 0 & 2n \\ \frac{2(-1)^{n+1}}{n} & 3-n^2 \end{vmatrix}}{\begin{vmatrix} 3-n^2 & 2n \\ -2n & 3-n^2 \end{vmatrix}} = \frac{4(-1)^n}{(3-n^2)^2+4n^2} = \frac{4(-1)^n}{n^4-2n^2+9}$$

$$d_n = \frac{\begin{vmatrix} 3-n^2 & 0 \\ -2n & \frac{2(-1)^{n+1}}{n} \end{vmatrix}}{n^4-2n^2+9} = \frac{2(-1)^n(n^2-3)}{n(n^4-2n^2+9)}$$

Partikulær løsning:

$$y = \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^4-2n^2+9} \cos nt + \frac{2(-1)^n(n^2-3)}{n(n^4-2n^2+9)} \sin nt \right)$$

17.5

11 $$\mathcal{F}\{x \operatorname{sgn} x\} = i\left(-i\sqrt{\tfrac{2}{\pi}}\right)\left(-\tfrac{1}{k^2}\right) = -\sqrt{\tfrac{2}{\pi}}\,\tfrac{1}{k^2}$$

12 $$f(t) = e^{2it}\,e^{-|t|}$$

$$\hat{F}(\omega) = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{(\omega-2)^2+1} = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{\omega^2-4\omega+5}$$

14 $$\mathcal{F}\{x\,e^{-a|x|}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\,e^{-a|x|}\,e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\,2\int_0^{\infty} x\,e^{-ax}(-i\sin kx)\,dx$$

$$= -i\sqrt{\tfrac{2}{\pi}}\,\frac{2ak}{(k^2+a^2)^2}$$ jf. integraltabel CRC 659

21 $$W(r) = \frac{1}{r} - \frac{\kappa^2}{4\pi}\int_{R^3}\frac{W(\bar{r}-\bar{r}')}{|\bar{r}'|}\,d\bar{r}'$$

$$\hat{W}(k) = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2} - \frac{\kappa^2}{4\pi}\sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2}(2\pi)^{\frac{3}{2}}\hat{W}(k) \qquad *$$

$$= \sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2} - \frac{\kappa^2}{k^2}\hat{W}(k)$$

$$\hat{W}(k) = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2}\,\frac{k^2}{k^2+\kappa^2} = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2+\kappa^2}$$

$$W(r) = \frac{1}{r}\,e^{-\kappa r}$$

$*$ $\mathcal{F}\{\tfrac{1}{r}\} = \sqrt{\tfrac{2}{\pi}}\,\tfrac{1}{k^2}$ jf. opg. 17 og 18

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{R^3}\frac{1}{|\bar{r}'|}\,W(\bar{r}-\bar{r}')\,d\bar{r}' = \sqrt{\tfrac{2}{\pi}}\,\frac{1}{k^2}\,\hat{W}(k)$$

(foldningsregel)

17.6

1 $\quad D \dfrac{\partial^2 c}{\partial x^2} = \dfrac{\partial c}{\partial t}$, beg. bet.: $c(x,0) = c_0 H(x)$

transf.: $\hat{c}(k,0) = c_0 \frac{1}{2}\left(\sqrt{2\pi}\,\delta(k) - i\sqrt{\frac{2}{\pi}}\frac{1}{k}\right)$

transf. løsn: $\hat{C}(k,t) = \frac{c_0}{2}\left(\sqrt{2\pi}\,\delta(k) - i\sqrt{\frac{2}{\pi}}\frac{1}{k}\right)e^{-k^2Dt}$

$$c(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{c_0}{2}\left(\sqrt{2\pi}\,\delta(k) - i\sqrt{\frac{2}{\pi}}\frac{1}{k}\right)e^{-k^2Dt}e^{ikx}dk$$

$$= \frac{c_0}{2}\left(\int_{-\infty}^{\infty}\delta(k)e^{-k^2Dt}e^{ikx}dk - i\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{k}e^{-k^2Dt}e^{ikx}dk\right.$$

$$= \frac{c_0}{2}\left(1 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(-i\sqrt{\frac{2}{\pi}}\frac{1}{k}\right)e^{-k^2Dt}e^{ikx}dk\right.$$

$$= \frac{c_0}{2}\left(1 + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\operatorname{sgn}(x-u)\frac{1}{\sqrt{2Dt}}e^{-\frac{u^2}{4Dt}}du\right) \quad *$$

$$= \frac{c_0}{2}\left(1 + \frac{1}{\sqrt{2\pi}\sqrt{2Dt}}\left(\int_{-\infty}^{x}e^{-\frac{u^2}{4Dt}}du - \int_{x}^{\infty}e^{-\frac{u^2}{4Dt}}du\right)\right.$$

$$= \frac{c_0}{2}\left(1 + \frac{1}{2\sqrt{\pi Dt}}\int_{-x}^{x}e^{-\frac{u^2}{4Dt}}du\right)$$

$$= \frac{c_0}{2}\left(1 + \frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-\frac{u^2}{4Dt}}d\left(\frac{u}{2\sqrt{Dt}}\right)\right)$$

$$= \frac{c_0}{2}\left(1 + \operatorname{erf}\frac{x}{2\sqrt{Dt}}\right)$$

$*$ $\quad \mathcal{F}^{-1}\left(\hat{f}(k)\,\hat{g}(k)\right) = \frac{1}{\sqrt{2\pi}}\, f * g\,(x)$ benyttet

6 $\quad$ Bemærk, at $\quad k_j e^{-i\bar{k}\cdot\bar{r}} = i\,\dfrac{\partial e^{-i\bar{k}\cdot\bar{r}}}{\partial x_j}$,

og at $k_j\, \mathcal{F}\{u(\bar{r},t)\} = \dfrac{k_j}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3} u(\bar{r},t)\,e^{-i\bar{k}\cdot\bar{r}}dV$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} u(\bar{r},t)\; i\,\frac{\partial e^{-i\bar{k}\cdot\bar{r}}}{\partial x_j}dV$$

$$= \frac{i}{(2\pi)^{\frac{3}{2}}}\left(\int_{\mathbb{R}^2}\left[u(\bar{r},t)e^{-i\bar{k}\cdot\bar{r}}\right]_{x_j=-\infty}^{x_j=\infty}dA\right.$$

$$\left. - \int_{\mathbb{R}^3}\frac{\partial u(\bar{r},t)}{\partial x_j}e^{-i\bar{k}\cdot\bar{r}}dV\right)$$

$$= 0 - i\,\mathcal{F}\left\{\frac{\partial u(\bar{r},t)}{\partial x_j}\right\}$$

fortsættes

17.6

6 fortsat

heraf $k_j^2 \; \mathcal{F}\{u(\bar r, t)\} = (-i)^2 \; \mathcal{F}\left\{ \frac{\partial^2 u(\bar r, t)}{\partial x_j^2} \right\}$

$$= - \mathcal{F}\left\{ \frac{\partial^2 u(\bar r, t)}{\partial x_j^2} \right\}$$

$$\Rightarrow \; \mathcal{F}\{\nabla^2 u(\bar r, t)\} = - |\bar k|^2 \; \mathcal{F}\{\bar u(\bar r, t)\}$$

$$= - k^2 \, \hat C(\bar k, t)$$