Notes for the course "Functional Analysis". I.

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1 Compact (kompakt) and sequentially compact (følgekompakt) sets

Definition 1.1. *Let A be a subset of a metric space (X, d). Let $\mathcal{F}$ be an arbitrary set of indices, and consider the family of sets $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$, where each $\mathcal{O}_\alpha \subseteq X$ is open. This family is called an open covering of A if $A \subseteq \bigcup_{\alpha\in\mathcal{F}} \mathcal{O}_\alpha$.*

Definition 1.2. *Assume that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$ is an open covering of A. If $\mathcal{F}'$ is a subset of $\mathcal{F}$, we say that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}'}$ is a subcovering if we still have the property $A \subseteq \bigcup_{\alpha\in\mathcal{F}'} \mathcal{O}_\alpha$. A subcovering is called finite, if $\mathcal{F}'$ contains finitely many elements.*

Definition 1.3. *Let A be a subset of a metric space (X, d). Then we say that A is covered by a finite ϵ-net if there exists a natural number $N_\epsilon < \infty$ and the points $\{\mathbf{x}_1, ..., \mathbf{x}_{N_\epsilon}\} \subseteq A$ such that $A \subseteq \bigcup_{j=1}^{N_\epsilon} B_\epsilon(\mathbf{x}_j)$.*

Definition 1.4. *A subset $A \subset X$ is called compact, if from* ANY *open covering of A one can extract a* FINITE *subcovering.*

Definition 1.5. *$A \subset X$ is called sequentially compact if from any sequence $\{x_n\}_{n\geq1} \subseteq A$ one can extract a subsequence $\{x_{n_k}\}_{k\geq1}$ which converges to some point $x_\infty \in A$.*

We will see that in metric spaces the two notions of compactness are equivalent.

1.1 Compact implies sequentially compact

Theorem 1.6. *Assume that $A \subseteq X$ is compact. Then A is sequentially compact.*

Proof. Assume that there exists a sequence $\{x_n\}_{n\geq1}$ with no convergent subsequence in A. Such a sequence must have an infinite number of distinct points (exercise). To give a hint, assume that the range of this sequence is $\{a, b\}$. If there only exist a finite number of points in the sequence which are equal with a, then there must exist an infinite number of points which are equal with b. These points would thus define a convergent subsequence, contradicting our hypothesis.

Therefore we can assume that $\{x_n\}_{n\geq1}$ has no accumulation points in A (otherwise such a point would be the limit of a subsequence). Now choose an arbitrary point $x \in A$. Because x is not an accumulation point for $\{x_n\}_{n\geq1}$, there exists $\epsilon_x > 0$ such that the ball $B_{\epsilon_x}(x)$ contains at most one element of $\{x_n\}_{n\geq1}$.

Because $\{B_{\epsilon_x}(x)\}_{x\in A}$ is an open covering for A, and since A is compact, we can extract a finite subcovering from it:

$$A \subseteq \bigcup_{j=1}^{N} B_{\epsilon_{x_j}}(x_j), \quad N < \infty.$$

But since $\{x_n\}_{n\geq 1} \subseteq A$, and because we know that there are at most N distinct points of this sequence in the union $\bigcup_{j=1}^N B_{\epsilon_{x_j}}(x_j)$, we conclude that $\{x_n\}_{n\geq 1}$ can only have a finite number of distinct points, thus it must admit a convergent subsequence. This contradicts our hypothesis. $\square$

1.2 Sequentially compact implies compact

The proof of this fact is slightly more complicated. We need a preparatory result:

Proposition 1.7. *Let A be a sequentially compact set. Then for every $\epsilon > 0$, A can be covered by a finite ϵ-net (see Definition 1.3).*

Proof. If A contains finitely many points, then the proof is obvious. Thus we assume $\#(A) = \infty$.

Now assume that there exists some $\epsilon_0 > 0$ such that A cannot be covered by a finite ϵ_0-net. This means that for any N points of A, $\{x_1, ..., x_N\}$, we have:

$$A \not\subset \bigcup_{j=1}^{N} B_{\epsilon_0}(x_j). \tag{1.1}$$

We will now construct a sequence with elements in A which cannot have a convergent subsequence. Choose an arbitrary point $x_1 \in A$. We know from (1.1), for $N = 1$, that we can find $x_2 \in A$ such that $x_2 \in A \setminus B(x_1, \epsilon_0)$. This means that $d(x_1, x_2) \geq \epsilon_0$. We use (1.1) again, for $N = 2$, in order to get a point $x_3 \in A \setminus [B(x_1, \epsilon_0) \cup B(x_2, \epsilon_0)]$. This means that $d(x_3, x_1) \geq \epsilon_0$ and $d(x_3, x_2) \geq \epsilon_0$. Thus we can continue with this procedure and construct a sequence $\{x_n\}_{n\geq 1} \subseteq A$ which obeys

$$d(x_j, x_k) \geq \epsilon_0, \quad j \neq k.$$

In other words, we constructed a sequence in A which consists only from isolated points, and which cannot have a convergent subsequence. This contradicts Definition 1.5. $\square$

Let us now prove the theorem:

Theorem 1.8. *Assume that $A \subseteq X$ is sequentially compact. Then A is compact.*

Proof. Consider an arbitrary open covering of A:

$$A \subseteq \bigcup_{\alpha \in \mathcal{F}} \mathcal{O}_\alpha.$$

We will show that we can extract a finite subcovering from it.

For every $x \in A$, there exists at least one open set $\mathcal{O}_{\alpha(x)}$ such that $x \in \mathcal{O}_{\alpha(x)}$. Because $\mathcal{O}_{\alpha(x)}$ is open, we can find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{O}_{\alpha(x)}$.

For a fixed x, we consider the supremum over all radii $\epsilon > 0$ which obey the condition that there exists at least one $\alpha \in \mathcal{F}$ such that $B_\epsilon(x) \subseteq \mathcal{O}_\alpha$. This supremum is larger than zero, since there exists at least one positive such ϵ. Now write this supremum as $2\epsilon_x > 0$. It means that if we take $\epsilon' > 2\epsilon_x$, then for every $\alpha \in \mathcal{F}$ we have $B_{\epsilon'}(x) \not\subseteq \mathcal{O}_\alpha$.

Let us write an important relation:

$$A \subseteq \bigcup_{x \in A} B_{\epsilon_x}(x) \subseteq \bigcup_{\alpha \in \mathcal{F}} \mathcal{O}_\alpha. \tag{1.2}$$

The first inclusion is obvious, while the second one follows from the above discussion.

We now need to prove a lemma:

Lemma 1.9. *If A is sequentially compact, then*

$$\inf_{x \in A} \epsilon_x =: 2\epsilon_0 > 0.$$

In other words, there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq B_{\epsilon_x}(x)$, for every $x \in A$.

Proof. Assume that $\inf_{x \in A} \epsilon_x = 0$. This implies that there exists a sequence $\{x_n\}_{n \geq 1} \subseteq A$ such that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. Since A is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k \geq 1}$ which converges to a point $x_0 \in A$, i.e.

$$\lim_{k \to \infty} x_{n_k} = x_0. \tag{1.3}$$

Because x_0 belongs to A, we can find an open set $\mathcal{O}_{\alpha(x_0)}$ which contains x_0, thus we can find $\epsilon_1 > 0$ such that

$$B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}. \tag{1.4}$$

Now (1.3) implies that there exists $K > 0$ large enough such that:

$$d(x_{n_k}, x_0) \leq \epsilon_1/4, \quad \text{whenever} \quad k > K. \tag{1.5}$$

If y belongs to $B_{\epsilon_1/4}(x_{n_k})$ (i.e. $d(y, x_{n_k}) < \epsilon_1/4$), then the triangle inequality implies (use also (1.5)):

$$d(y, x_0) \leq d(y, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon_1/2 < \epsilon_1, \quad k > K.$$

But this shows that we must have $y \in B_{\epsilon_1}(x_0)$, or:

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K. \tag{1.6}$$

Thus we got the inclusion

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K,$$

which shows that $\epsilon_1/4$ must be less or equal than $2\epsilon_{x_{n_k}}$, or $\epsilon_1/8 \leq \epsilon_{x_{n_k}}$, for every $k > K$. But this is in contradiction with the fact that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. □

Finishing the proof of Theorem 1.8. We now use Proposition 1.7, and find a finite ϵ_0-net for A. Thus we can choose $\{y_1, ...y_N\} \subseteq A$ such that

$$A \subseteq \bigcup_{n=1}^{N} B_{\epsilon_0}(y_n) \subseteq \bigcup_{n=1}^{N} B_{\epsilon_{y_n}}(y_n) \subseteq \bigcup_{n=1}^{N} \mathcal{O}_n,$$

where $\mathcal{O}_n$ is one of the possibly many other open sets which contain $B_{\epsilon_{y_n}}(y_n)$. We have thus extracted our finite subcovering of A and the proof of the theorem is over. □