Power series are analytic

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1 Fubini's theorem for double series

Theorem 1.1. *Let* $\{\alpha_{nm}\}_{n,m\geq 0}$ *be a real sequence indexed by two indices. Assume that the series* $\sum_{m\geq 0}|\alpha_{nm}|$ *is convergent for all* n *and*

$$C := \sum_{n\geq 0}\Big(\sum_{m\geq 0}|\alpha_{nm}|\Big) < \infty. \tag{1.1}$$

Then we have that $\sum_{n\geq 0}|\alpha_{nm}|$ *converges for all* m *and:*

$$\sum_{m\geq 0}\Big(\sum_{n\geq 0}|\alpha_{nm}|\Big) = C. \tag{1.2}$$

Moreover,

$$\lim_{N\to\infty}\sum_{m\geq 0}\Big(\sum_{n>N}|\alpha_{nm}|\Big) = \lim_{M\to\infty}\sum_{n\geq 0}\Big(\sum_{m>M}|\alpha_{nm}|\Big) = 0. \tag{1.3}$$

Finally,

$$\sum_{m\geq 0}\Big(\sum_{n\geq 0}\alpha_{nm}\Big) = \sum_{n\geq 0}\Big(\sum_{m\geq 0}\alpha_{nm}\Big) \in \mathbb{R}. \tag{1.4}$$

Proof. We recall a few fundamental results. If $a_n \geq 0$ is a nonnegative sequence, we define $s_N = \sum_{n=0}^{N} a_n$ to be an increasing sequence of partial sums. Then $\sum_{n\geq 0} a_n = \lim_{N\to\infty} s_N$ exists and is finite if and only if the sequence $\{s_N\}_{N\geq 0}$ is bounded from above. Moreover, if s_N converges then it is Cauchy, hence for all $\epsilon > 0$ there exists $N_\epsilon \geq 0$ such that $0 \leq s_{N+k} - s_N < \epsilon$ for all $k \geq 1$ and $N \geq N_\epsilon$. This implies:

$$0 \leq s_{N+k} - s_N = \sum_{n=N+1}^{N+k} a_n < \epsilon, \quad \forall k \geq 1.$$

Taking the supremum over k we get $0 \leq \sum_{n\geq N+1} a_n \leq \epsilon$ for every $N \geq N_\epsilon$. In other words:

$$\lim_{N\to\infty}\sum_{n>N} a_n = 0. \tag{1.5}$$

If N and M are finite natural numbers, then we have:

$$\sum_{m=0}^{M}\sum_{n=0}^{N}|\alpha_{nm}| = \sum_{n=0}^{N}\sum_{m=0}^{M}|\alpha_{nm}| \leq \sum_{n=0}^{N}\sum_{m\geq 0}|\alpha_{nm}| \leq C. \tag{1.6}$$

In the last two inequalities we employed the assumption (1.1). Hence

$$\sum_{m=0}^{M}\sum_{n=0}^{N}|\alpha_{nm}| \leq C < \infty, \quad \forall N, M \geq 0. \tag{1.7}$$

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In particular,

$$\sum_{n=0}^{N} |\alpha_{nm}| \leq C < \infty, \quad \forall N, m \geq 0.$$

This shows that $\sum_{n\geq 0} |\alpha_{nm}|$ is convergent for all $m \geq 0$. Now we can take the limit $N \to \infty$ in (1.7) and obtain:

$$\sum_{m=0}^{M} \sum_{n\geq 0} |\alpha_{nm}| \leq C < \infty, \quad \forall M \geq 0.$$

But this shows that the sequence of the partial sums generated by $a_m := \sum_{n\geq 0} |\alpha_{nm}|$ is bounded, hence

$$D := \sum_{m\geq 0} \big(\sum_{n\geq 0} |\alpha_{nm}| \big) \leq C.$$

Now using again the first identity in (1.6) we have:

$$\sum_{n=0}^{N} \sum_{m=0}^{M} |\alpha_{nm}| = \sum_{m=0}^{M} \sum_{n=0}^{N} |\alpha_{nm}| \leq \sum_{m=0}^{M} \sum_{n\geq 0} |\alpha_{nm}| \leq D$$

or

$$\sum_{n=0}^{N} \sum_{m=0}^{M} |\alpha_{nm}| \leq D, \quad \forall N, M \geq 0.$$

Our hypothesis guarantees that $\lim_{M\to\infty} \sum_{m=0}^{M} |\alpha_{nm}|$ exists and is finite, hence:

$$\sum_{n=0}^{N} \sum_{m\geq 0} |\alpha_{nm}| \leq D, \quad \forall N \geq 0.$$

Thus by taking $N \to \infty$ we get:

$$C = \sum_{n\geq 0} \sum_{m\geq 0} |\alpha_{nm}| \leq D$$

which proves that $C = D$.

Now we have to prove (1.3). Define $\beta_{nm} = \alpha_{nm}$ if $n > N$, and $\beta_{nm} = 0$ if $0 \leq n \leq N$. Then we have:

$$\sum_{m\geq 0} \sum_{n\geq 0} |\beta_{nm}| = \sum_{n\geq 0} \sum_{m\geq 0} |\beta_{nm}| \quad \text{or} \quad \sum_{m\geq 0} \sum_{n>N} |\alpha_{nm}| = \sum_{n>N} \big(\sum_{m\geq 0} |\alpha_{nm}| \big).$$

Denoting by $a_n = \sum_{m\geq 0} |\alpha_{nm}|$ we see that (use (1.5)):

$$\sum_{m\geq 0} \sum_{n>N} |\alpha_{nm}| = \sum_{n>N} a_n \to 0 \quad \text{when} \quad N \to \infty.$$

In a similar way we can prove the other limit in (1.3).

Now we have to prove (1.4). First of all, because

$$\big| \sum_{m\geq 0} \alpha_{nm} \big| \leq \sum_{m\geq 0} |\alpha_{nm}|, \quad \forall n \geq 0$$

we have that $\sum_{n\geq 0} \big(\sum_{m\geq 0} \alpha_{nm} \big)$ is absolutely convergent. The same holds true for the series in the right hand side of (1.4). Thus we only need to prove that the two double series are equal.

If N and M are finite natural numbers we have:

$$\sum_{m=0}^{M} \sum_{n=0}^{N} \alpha_{nm} = \sum_{n=0}^{N} \sum_{m=0}^{M} \alpha_{nm}, \tag{1.8}$$

which implies:

$$\sum_{m=0}^{M}\sum_{n\geq 0}\alpha_{nm}-\sum_{n=0}^{N}\sum_{m\geq 0}\alpha_{nm}=\sum_{m=0}^{M}\sum_{n>N}\alpha_{nm}-\sum_{n=0}^{N}\sum_{m>M}\alpha_{nm}, \tag{1.9}$$

which leads to:

$$\left|\sum_{m=0}^{M}\sum_{n\geq 0}\alpha_{nm}-\sum_{n=0}^{N}\sum_{m\geq 0}\alpha_{nm}\right|\leq\sum_{m\geq 0}\sum_{n>N}|\alpha_{nm}|+\sum_{n\geq 0}\sum_{m>M}|\alpha_{nm}|. \tag{1.10}$$

Now we use (1.3) in (1.10): take both M and N to infinity, and obtain:

$$\left|\sum_{m\geq 0}\sum_{n\geq 0}\alpha_{nm}-\sum_{n\geq 0}\sum_{m\geq 0}\alpha_{nm}\right|\leq 0$$

which ends the proof.

□

2 The binomial identity

Theorem 2.1. *Let $a, b\in\mathbb{R}$ and $n\in\mathbb{N}$. Then:*

$$(a+b)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}a^k b^{n-k}.$$

Proof. Let $P:\mathbb{R}\mapsto\mathbb{R}$ given by $P(x)=(x+b)^n$. We have that $P'(x)=n(x+b)^{n-1}$, $P''(x)=n(n-1)(x+b)^{n-2}$, and by induction we can prove:

$$P^{(k)}(x)=n(n-1)\dots(n-k+1)(x+b)^{n-k}=\frac{n!}{(n-k)!}(x+b)^{n-k},\quad 0\leq k\leq n.$$

Moreover, $P^{(k)}(x)=0$ if $k>n$. The Taylor formula with remainder provides us with some c between 0 and x such that:

$$P(x)=P(0)+\sum_{k=1}^{n}\frac{P^{(k)}(0)}{k!}x^k+\frac{P^{(n+1)}(c)}{(n+1)!}x^k=P(0)+\sum_{k=1}^{n}\frac{P^{(k)}(0)}{k!}x^k.$$

The final result is obtained by replacing x with a. □

3 The exponential and the logarithm

For every $x\in\mathbb{R}$ we define

$$\exp(x):=1+x+\frac{x^2}{2}+\dots+\frac{x^n}{n!}+\dots=\sum_{n\geq 0}\frac{x^n}{n!},$$

assuming for the moment that it converges. If $x=0$ we have $\exp(0)=1$. If $x\neq 0$ consider the series $\sum_{n\geq 0}\alpha_n$ given by $\alpha_n=\frac{x^n}{n!}$. Using the ratio criterion we have:

$$\frac{|\alpha_{n+1}|}{|\alpha_n|}=\frac{|x|}{n+1}\to 0<1\quad\text{when}\quad n\to\infty,$$

which shows that the series defining $\exp(x)$ converges absolutely.

We want to prove that the exponential is everywhere differentiable. Fix $a \in \mathbb{R}$ and let $h \in \mathbb{R}$. Define the function

$$F(h) := (h+a)^n, \quad n \geq 2. \tag{3.11}$$

The Taylor formula with remainder provides us with a $c = c_{n,a,h}$ between 0 and h such that $F(h) = F(0) + F'(0)h + F''(c)h^2/2$, or:

$$(h+a)^n - a^n = na^{n-1}h + \frac{n(n-1)}{2}h^2(c_{n,a,h}+a)^{n-2}, \tag{3.12}$$

which leads to:

$$\frac{(h+a)^n}{n!} - \frac{a^n}{n!} = \frac{a^{n-1}}{(n-1)!}h + \frac{(c_{n,a,h}+a)^{n-2}}{2(n-2)!}h^2, \quad n \geq 2. \tag{3.13}$$

Thus if $h \neq 0$:

$$\frac{\exp(h+a) - \exp(a)}{h} = 1 + \sum_{n\geq 2} \frac{a^{n-1}}{(n-1)!} + \frac{h}{2}\sum_{n\geq 2}\frac{(c_{n,a,h}+a)^{n-2}}{(n-2)!}.$$

Note first that $1 + \sum_{n\geq 2} \frac{a^{n-1}}{(n-1)!} = \exp(a)$. Moreover, since $|c_{n,a,h} + a| \leq |a| + |h|$ we may write:

$$\left|\frac{\exp(h+a) - \exp(a)}{h} - \exp(a)\right| \leq \frac{|h|}{2}\sum_{n\geq 2}\frac{(|a|+|h|)^{n-2}}{(n-2)!} = \frac{|h|\exp(|a|+|h|)}{2} \leq \frac{|h|\exp(|a|+1)}{2},$$

which holds for every $0 < |h| \leq 1$. It follows that the exponential function is differentiable at a and $\exp'(a) = \exp(a)$.

Theorem 3.1. *We have that* $\exp(-x)\exp(x) = 1$ *and* $\exp(x) > 0$ *for all* $x \in \mathbb{R}$*. Moreover,* $\exp(a+b) = \exp(a)\exp(b)$ *for all* $a, b \in \mathbb{R}$*. Define the logarithm function*

$$\ln(x) := \int_1^x \frac{1}{t}dt, \quad x > 0.$$

Then we have $\ln(\exp(x)) = x$ *for all* $x \in \mathbb{R}$*, and* $\exp(\ln(x)) = x$ *for all* $x > 0$*.*

Proof. We know that $\exp(0) = 1$ and $\exp'(x) = \exp(x)$ holds on $\mathbb{R}$. Define the function $f(x) = \exp(-x)\exp(x)$. Then f is differentiable and

$$f(0) = 1, \quad f'(x) = 0, \quad \forall x \in \mathbb{R}.$$

Hence $f(x) = 1$ on $\mathbb{R}$, which proves that $\exp(-x)\exp(x) = 1$ for all $x \in \mathbb{R}$. The same identity shows that $\exp(x)$ can never be zero. Now since $\exp(0) = 1 > 0$ and because exp is continuous (being differentiable), it cannot change sign because it would have to go through a zero (remember the intermediate value theorem). Hence $\exp(x) > 0$ on $\mathbb{R}$.

Now define the function $g(x) = \exp(-x-b)\exp(x)\exp(b)$ for some fixed b. We again have $g(0) = 1$ and $g'(x) = 0$ for all $x \in \mathbb{R}$, hence $\exp(-x-b)\exp(x)\exp(b) = 1$ on $\mathbb{R}$. Multiply with $\exp(x+b)$ on both sides and obtain $\exp(x)\exp(b) = \exp(x+b)$ on $\mathbb{R}$.

The logarithm function is defined to be a primitive of $1/x$, i.e.:

$$\ln'(x) = \frac{1}{x}, \quad \ln(1) = 0.$$

Define $f(x) = \ln(\exp(x)) - x$ on $\mathbb{R}$, which is possible because $\exp(x) > 0$. We have $f(0) = 0$ and $f'(x) = 0$ for all $x \in \mathbb{R}$, hence $\ln(\exp(x)) = x$ on $\mathbb{R}$.

If $x > 0$, consider the function $f(x) = \frac{1}{x}\exp(\ln(x))$. We have that $f(1) = 1$ and $f'(x) = 0$ for all $x > 0$, hence $\exp(\ln(x)) = x$ for all $x > 0$.

We have just proved that the exponential and the logarithm are inverses to each other. $\square$

Corollary 3.2. *We have* $\ln(ab) = \ln(a) + \ln(b)$ *for all* $a, b > 0$. *Moreover,* $\ln(y^x) = x\ln(y)$ *for all* $y > 0$ *and* $x \in \mathbb{R}$. *Thus if* $y > 0$ *and* $x \in \mathbb{R}$, *we have* $y^x = \exp(x\ln(y))$.

Proof. Since
$$\exp(\ln(ab)) = ab = \exp(\ln(a))\exp(\ln(b)) = \exp(\ln(a) + \ln(b)),$$
we must have $\ln(ab) = \ln(a) + \ln(b)$ due to the injectivity of exp. If $ab = 1$ we have $0 = \ln(a) + \ln(a^{-1})$, or $\ln(a^{-1}) = -\ln(a)$. Now if $a = b$ we get $\ln(a^2) = 2\ln(a)$. By induction, we obtain that $\ln(a^n) = n\ln(a)$ for all $n \in \mathbb{N}$. Replacing a in the last identity with $b^{1/n}$ we obtain $\ln(b^{1/n}) = \frac{1}{n}\ln(b)$. Thus $\ln(b^{\frac{m}{n}}) = \frac{m}{n}\ln(b)$. Moreover, $\ln(b^{-\frac{m}{n}}) = -\frac{m}{n}\ln(b)$.

Thus we have just proved that for every rational number r and for every positive number $y > 0$ we have $\ln(y^r) = r\ln(y)$. This implies $y^r = \exp(r\ln(y))$ for every rational number r. Finally, we use that every real number x is the limit of a sequence of rational numbers, together with the continuity of exp. □

Corollary 3.3. *Let* $\alpha, \beta, \gamma > 0$. *We have that*
$$\lim_{x\to\infty} \frac{x^\alpha}{\exp(\beta x)} = \lim_{x\to\infty} \frac{\ln(x)}{x^\gamma} = 0. \tag{3.14}$$

Proof. Let N be an integer such that $\alpha < N$. We have the inequality:
$$\exp(\beta x) \geq 1 + \beta x + \cdots + \frac{\beta^N x^N}{N!} \geq \frac{\beta^N x^N}{N!}, \quad \forall x > 0.$$
Then:
$$0 \leq \frac{x^\alpha}{\exp(\beta x)} \leq \frac{N!}{\beta^N x^{N-\alpha}} \to 0 \quad \text{when} \quad x \to \infty.$$

Now if $\gamma > 0$ and $x > 0$ we have $x^\gamma = \exp(\gamma\ln(x))$. Denote by $y = \ln(x)$. Then we have:
$$\lim_{x\to\infty} \frac{\ln(x)}{x^\gamma} = \lim_{y\to\infty} \frac{y}{\exp(\gamma y)} = 0.$$
□

4 Power series are analytic functions

Let $\{a_n\}_{n\geq 0} \subset \mathbb{R}$ such that $\limsup_{n\to\infty} |a_n|^{1/n} < \infty$. Define $r = 1/\{\limsup_{n\to\infty} |a_n|^{1/n}\}$ if $\limsup_{n\to\infty} |a_n|^{1/n} > 0$ and $r = \infty$ if $\limsup_{n\to\infty} |a_n|^{1/n} = 0$.

Let $0 < R < r$ and define $f : (x_0 - R, x_0 + R) \mapsto \mathbb{R}$ given by:
$$f(x) := \sum_{n\geq 0} a_n (x - x_0)^n.$$

The series is absolutely convergent because $\limsup_{n\to\infty} |a_n(x - x_0)^n|^{1/n} = \frac{|x-x_0|}{r} < 1$.

Theorem 4.1. *Let* $b \in (x_0 - R, x_0 + R)$ *be an arbitrary point. Then* f *is indefinitely differentiable at* b, *and for every* $t \in (x_0 - R, x_0 + R)$ *with* $|t - b| < R - |b - x_0|$ *we have:*
$$f(t) = \sum_{m\geq 0} \frac{f^{(m)}(b)}{m!}(t - b)^m,$$
where the Taylor series is absolutely convergent.

Proof. Denote by $\alpha_{nm} := n(n-1)\dots(n-m+1)a_n$ if $m \geq 1$. Note that if $n > k$ we have:

$$(n-k)^{1/n} = \exp(\ln[(n-k)^{1/n}]) = \exp\left(\frac{\ln(n-k)}{n}\right) = \exp\left(\frac{\ln(n) + \ln(1-k/n)}{n}\right)$$

and using (3.14):

$$\exp\left(\frac{\ln(n)}{n} + \frac{\ln(1-k/n)}{n}\right) \to \exp(0) = 1 \quad \text{when} \quad n \to \infty.$$

It follows that

$$\limsup_{n\to\infty} |\alpha_{nm}|^{1/n} = \frac{1}{r}, \quad \forall m \geq 1.$$

Thus the series $\sum_{n\geq m} \alpha_{nm} t^{n-m}$ is absolutely convergent for all $|t| < R$. Given x such that $|x - x_0| \leq \rho < R < r$, there exists some $h_0 > 0$ such that $|x + h - x_0| \leq (R+\rho)/2 < R$ for all $|h| \leq h_0$. Using (3.12) with $a = x - x_0$ and $|h| \leq h_0$ we have:

$$f(x+h) - f(x) = h \sum_{n\geq 1} n a_n (x - x_0)^{n-1} + \frac{h^2}{2} \sum_{n\geq 2} n(n-1) a_n (x + c_{n,a,h} - x_0)^{n-2},$$

where $c_{n,a,h}$ lies between 0 and h. Note that both series on the right hand side converge absolutely because:

$$|n a_n (x - x_0)^{n-1}| \leq |\alpha_{n1}| \rho^{n-1}, \quad |n(n-1) a_n (x + c_{n,a,h} - x_0)^{n-2}| \leq |\alpha_{n2}| [(R+\rho)/2]^{n-2}.$$

We conclude that $f'(x) = \sum_{n\geq 1} n a_n (x - x_0)^{n-1}$ for all $|x - x_0| < R$. By induction, we obtain:

$$f^{(m)}(x) = \sum_{n\geq m} \alpha_{nm} (x - x_0)^{n-m}, \quad m \geq 1.$$

It follows that we have the identity:

$$\frac{f^{(m)}(x)}{m!} h^m = \sum_{n\geq m} a_n \frac{n!}{(n-m)! m!} h^m (x - x_0)^{n-m}$$

which holds true for all $m \geq 0$.

Now define $\beta_{nm} = 0$ if $m > n$ and $\beta_{nm} = a_n \frac{n!}{(n-k)!k!} h^m (x - x_0)^{n-m}$ if $m \leq n$. We see that

$$\sum_{m\geq 0} |\beta_{nm}| = \sum_{m=0}^{n} |\beta_{nm}| \leq |a_n| \sum_{m=0}^{n} \frac{n!}{(n-m)! m!} |h|^m |x - x_0|^{n-m} = |a_n| (|h| + |x - x_0|)^n$$

where we used the binomial identity in the last equality. Now if $|h| < R - |x - x_0|$ it follows that $\sum_{n\geq 0} |a_n| (|h| + |x - x_0|)^n < \infty$, hence:

$$\sum_{n\geq 0} \sum_{m\geq 0} |\beta_{nm}| < \infty.$$

The conditions of Theorem 1.1 are satisfied, hence

$$\sum_{n\geq 0} \sum_{m\geq 0} \beta_{nm} = \sum_{m\geq 0} \sum_{n\geq 0} \beta_{nm}.$$

Now we observe that

$$\sum_{n\geq 0} \sum_{m\geq 0} \beta_{nm} = \sum_{n\geq 0} \sum_{m=0}^{n} \beta_{nm} = \sum_{n\geq 0} a_n (x + h - x_0)^n = f(x+h),$$

while

$$\sum_{m\geq 0}\sum_{n\geq 0}\beta_{nm}=\sum_{m\geq 0}\sum_{n\geq m}\beta_{nm}=\sum_{m\geq 0}\sum_{n\geq m}a_n\frac{n!}{(n-m)!m!}h^m(x-x_0)^{n-m}=\sum_{m\geq 0}\frac{f^{(m)}(x)}{m!}h^m.$$

In other words,

$$f(x+h)=\sum_{m\geq 0}\frac{f^{(m)}(x)}{m!}h^m.$$

Now replace $x+h=t$ and $x=b$ and the theorem is proved. $\square$