Basic properties of limsup and liminf

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1 Equivalent definitions

Let $\{s_n\}_{n\geq 1}$ be a bounded real sequence, i.e. there exists $M > 0$ such that $-M \leq s_n \leq M$ for all $n \geq 1$. Then the sequence

$$\alpha_k := \sup\{s_n : n \geq k\} =: \sup_{n\geq k} s_n, \quad k \geq 1$$

is a decreasing sequence ($\alpha_{k+1} \leq \alpha_k$) bounded below by $-M$. Thus $\{\alpha_k\}_{k\geq 1}$ converges towards the infimum of its range. Therefore, we can define:

$$\limsup s_n := \lim_{k\to\infty} \alpha_k = \inf_{k\geq 1} \sup_{n\geq k} s_n. \tag{1.1}$$

For every $\epsilon > 0$ there exists k_ϵ such that

$$\limsup s_n \leq \alpha_k = \sup_{n\geq k} s_n \leq \alpha_{k_\epsilon} < (\limsup s_n) + \epsilon, \quad \forall k \geq k_\epsilon. \tag{1.2}$$

Similarly, the sequence

$$\beta_k := \inf\{s_n : n \geq k\} =: \inf_{n\geq k} s_n, \quad k \geq 1$$

is an increasing sequence ($\beta_{k+1} \geq \beta_k$) bounded above by M. Thus $\{\beta_k\}_{k\geq 1}$ converges towards the supremum of its range. Therefore, we can define:

$$\liminf s_n := \lim_{k\to\infty} \beta_k = \sup_{k\geq 1} \inf_{n\geq k} s_n. \tag{1.3}$$

For every $\epsilon > 0$ there exists k_ϵ such that

$$(\liminf s_n) - \epsilon < \beta_{k_\epsilon} \leq \inf_{n\geq k} s_n = \beta_k \leq \liminf s_n, \quad \forall k \geq k_\epsilon. \tag{1.4}$$

Let S denote the set of all real numbers for which there exists at least one subsequence $\{s_{n_j}\}_{j\geq 1}$ such that s_{n_j} converges to x when $j \to \infty$. Clearly, S is a subset of $[-M, M]$.

Theorem 1.1. *We have* $\max(S) = \limsup s_n$ *and* $\min(S) = \liminf s_n$.

Proof. We only show the first equality. In order to simplify notation we put $L := \limsup s_n$. There are two things we have to prove: (1) $\sup(S) \leq L$ and (2) $L \in S$. They would imply:

$$\sup(S) = \max(S) = L.$$

Let us start by proving (1). Assume that it is not true, i.e. $L < \sup(S)$. Because $\sup(S)$ is the smallest upper bound of S, it implies that L cannot be an upper bound, thus there exists some $x \in S$ such that $L < x \leq \sup(S)$. Since $x \in S$, there must exist a subsequence $\{s_{n_j}\}_{j\geq 1}$ such that $|s_{n_j} - x| \to 0$ when $j \to \infty$. In particular, there exists some $K \geq 1$ such that:

$$\frac{x + L}{2} < s_{n_j}, \quad \forall j \geq K. \tag{1.5}$$

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Formula (1.2) can be rewritten in the following way. For every $\epsilon > 0$, there exists k_ϵ large enough such that:

$$L \leq \sup_{n \geq k} s_n < L + \epsilon, \quad \forall k \geq k_\epsilon. \tag{1.6}$$

Now fix $\epsilon_0 = (x - L)/2$. The above inequality insures the existence of some K_0 such that:

$$s_n \leq \sup_{n \geq K_0} s_n < L + \epsilon_0 = \frac{x + L}{2}, \quad \forall n \geq K_0.$$

This inequality implies that only finitely many elements (at most K_0) of the sequence s_n can be larger than $\frac{x+L}{2}$, which contradicts (1.5). Thus (1) is proved.

Now let us prove (2) by constructing a subsequence $\{s_{n_j}\}_{j \geq 1}$ which has L as its limit. The idea is to find an increasing sequence $n_1 < n_2 < ... < n_j < ...$ such that

$$L - \frac{1}{j} \leq s_{n_j} \leq L + \frac{1}{j}, \quad \forall j \geq 1. \tag{1.7}$$

We will do this by induction, and the double inequality in (1.6) will play an important role. Let us start by constructing n_1. Let $\epsilon = 1$ in (1.6) and consider the corresponding k_1. We know that $(\sup_{n \geq k_1} s_n) - 1$ is not an upper bound for the set $\{s_n : n \geq k_1\}$, hence we may find $n_1 \geq k_1$ such that:

$$(\sup_{n \geq k_1} s_n) - 1 < s_{n_1}.$$

Using this in (1.6) with $k = k_1$ we get:

$$L - 1 \leq (\sup_{n \geq k_1} s_n) - 1 < s_{n_1} \leq \sup_{n \geq k_1} s_n < L + 1. \tag{1.8}$$

Now let us assume that (1.7) holds for $n_1 < ... < n_j$ and we want to construct n_{j+1}. Put $\epsilon = \frac{1}{j+1}$ in (1.6) and consider the corresponding $k_{\frac{1}{j+1}}$. Define:

$$\tilde{k}_j := \max\{n_j, k_{\frac{1}{j+1}}\} + 1.$$

Clearly, $n_j < \tilde{k}_j$ and $\tilde{k}_j > k_{\frac{1}{j+1}}$. In particular, we can apply (1.6) if $\epsilon = \frac{1}{j+1}$ and $k = \tilde{k}_j$ and we obtain:

$$L \leq \sup_{n \geq \tilde{k}_j} s_n < L + \frac{1}{j+1}.$$

Again, $(\sup_{n \geq \tilde{k}_j} s_n) - \frac{1}{j+1}$ is not an upper bound for the set $\{s_n : n \geq \tilde{k}_j\}$, hence we may find $n_{j+1} \geq \tilde{k}_j > n_j$ such that:

$$L - \frac{1}{j+1} \leq (\sup_{n \geq \tilde{k}_j} s_n) - \frac{1}{j+1} < s_{n_{j+1}} \leq \sup_{n \geq \tilde{k}_j} s_n < L + \frac{1}{j+1} \tag{1.9}$$

which finishes the induction step and proves (2). □

We end this section with two more definitions. If the sequence $\{s_n\}_{n \geq 1}$ is not bounded from above, then by convention we put $\limsup s_n = +\infty$. If the sequence is not bounded from below, we put $\liminf s_n = -\infty$.

2 Computation rules

Proposition 2.1. *Let* $\{s_n\}_{n\geq 1}$ *be a bounded real sequence. Then* $\liminf s_n \leq \limsup s_n$. *Moreover, the sequence is convergent and has the limit* L *if and only if* $\liminf s_n = \limsup s_n = L$.

Proof. From Theorem 1.1 we know that $\liminf s_n = \min(S) \leq \max(S) = \limsup s_n$.

Now let us prove the equivalence between convergence and equality of lim inf with lim sup. If the sequence is convergent to L, then we know that any subsequence can only converge to L. It follows that $S = \{L\}$, hence $\min(S) = \max(S) = L$.

The other way around: assume that $\liminf s_n = \limsup s_n = L$. Remember that for every $k \geq 1$ we have:

$$\alpha_k = \inf_{n\geq k} s_n \leq s_k \leq \sup_{n\geq k} s_n = \beta_k.$$

Since both α_k and β_k have the same limit L, we know that s_k also converges and has the same limit L. □

Proposition 2.2. *Let* $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be two bounded real sequences. Then we have*

$$\limsup(a_n + b_n) \leq \limsup(a_n) + \limsup(b_n) \quad \text{and} \quad \liminf(a_n) + \liminf(b_n) \leq \liminf(a_n + b_n).$$

Proof. For every $k \geq 1$ we can write:

$$\inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq a_j + b_j \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n, \quad \forall j \geq k.$$

The first inequality implies:

$$\inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq \inf_{n\geq k}(a_n + b_n),$$

while the second one gives:

$$\sup_{n\geq k}(a_n + b_n) \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n.$$

Now we can take $k \to \infty$ and we are done. □

Proposition 2.3. *Let* $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be two bounded real sequences such that* b_n *converges to* b. *Then:*

$$\limsup(a_n + b_n) = \limsup(a_n) + b \quad \text{and} \quad \liminf(a_n + b_n) = \liminf(a_n) + b.$$

Proof. We only prove the first identity, by showing a double inequality. From Proposition 2.2 we have:

$$\limsup(a_n + b_n) \leq \limsup(a_n) + \limsup(b_n) = \limsup(a_n) + b,$$

where in the second equality we used Proposition 2.1.

By writing $a_n = (a_n + b_n) + (-b_n)$ and use again Proposition 2.2 we have:

$$\limsup(a_n) \leq \limsup(a_n + b_n) + \limsup(-b_n) = \limsup(a_n + b_n) - b$$

where we used that $-b_n$ converges to $-b$. Hence:

$$\limsup(a_n) + b \leq \limsup(a_n + b_n),$$

and we are done.

□

Proposition 2.4. *Let* $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be two bounded real sequences such that* b_n *converges to* $b \geq 0$. *Then:*

$$\limsup(a_n b_n) = b \limsup(a_n) \quad \text{and} \quad \liminf(a_n b_n) = b \liminf(a_n).$$

Proof. We only prove the first identity. Remember the following elementary fact: if s_n is a bounded sequence and t_n converges to zero, then $s_n t_n$ converges to zero. This covers the case $b = 0$, thus we can assume $b > 0$. We have:

$$a_n b_n = a_n b + a_n(b_n - b).$$

Since $a_n(b_n - b)$ converges to zero, we can use Proposition 2.3 and obtain:

$$\limsup(a_n b_n) = \limsup(a_n b).$$

Because $b > 0$ we have $\sup_{n\geq k}(a_n b) = b \sup_{n\geq k}(a_n)$ and similarly:

$$\inf_{k\geq 1}(b \sup_{n\geq k}(a_n)) = b \inf_{k\geq 1} \sup_{n\geq k}(a_n).$$

Thus $\limsup(a_n b) = b \limsup(a_n)$ and we are done. $\square$