ODE exercises

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Exercise 1. Consider a function $\mathbf{f} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ where $\mathbf{f} \in C^1(\mathbb{R}^{d+1})$, which obeys the estimate

$$||\mathbf{f}(t,x)|| \leq C||\mathbf{x}||, \quad \forall [t,\mathbf{x}] \in \mathbb{R}^{d+1}.$$

Consider the equation $\mathbf{y}'(t) = \mathbf{f}(t,\mathbf{y}(t))$, where $\mathbf{y}(0) = \mathbf{y}_0$. Show that there exists a unique solution $\mathbf{y} : \mathbb{R} \mapsto \mathbb{R}^d$ which solves the equation for all $t \in \mathbb{R}$.

Hint. Use Lemma 6.3 in my notes in order to prove that $\mathbf{f}$ obeys a local Lipschitz condition on the set $[-1,1] \times \overline{B_1(\mathbf{y}_0)}$. Then one can apply Theorem 5.3 in my notes (local existence) and conclude that there exists a positive $\delta_1 > 0$ and a differentiable function $\mathbf{y} : (-\delta_1, \delta_1) \mapsto \mathbb{R}^d$ which is a solution to our ODE and also obeys:

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s,\mathbf{y}(s))ds, \quad |t| < \delta_1.$$

Using the estimate $||\mathbf{f}(s,\mathbf{y}(s))|| \leq C||\mathbf{y}(s)||$ we can write (take $t > 0$):

$$\begin{aligned}
||\mathbf{y}(t)|| &\leq ||\mathbf{y}(0)|| + C\int_0^t ||\mathbf{y}(s)||ds \leq ||\mathbf{y}(0)|| + Ct||\mathbf{y}(0)|| + C^2\int_0^t ds_1 \int_0^{s_1} ds_2||\mathbf{y}(s_2)|| \\
&\leq ||\mathbf{y}(0)|| + Ct||\mathbf{y}(0)|| + \frac{C^2t^2}{2}||\mathbf{y}(0)|| + C^3\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3||\mathbf{y}(s_3)|| \leq ... \\
&\leq ||\mathbf{y}(0)||e^{Ct}. \qquad (0.1)
\end{aligned}$$

Now assume that we cannot find a global in time solution, i.e. it only exists for a time interval of the form $(-T_1, T_2)$ where $T := \min\{T_1, T_2\} < \infty$ and $0 < \delta_1 < T$. Assume without loss that $T = T_2 < \infty$. We then have:

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s,\mathbf{y}(s))ds, \quad ||\mathbf{y}(t)|| \leq ||\mathbf{y}(0)||e^{CT}, \quad |t| < T.$$

Consider an arbitrary sequence $t_m \in (0,T)$ which converges to T. Then the sequence of values $\mathbf{y}(t_m) \in \mathbb{R}^d$ is Cauchy, because if $p > q$ we can write (assume without loss that $t_p > t_q$):

$$||\mathbf{y}(t_p) - \mathbf{y}(t_q)|| = ||\int_{t_q}^{t_p} \mathbf{f}(s,\mathbf{y}(s))ds|| \leq \int_{t_q}^{t_p} ||\mathbf{f}(s,\mathbf{y}(s))||ds \leq C||\mathbf{y}(0)||e^{CT}|t_p - t_q|$$

which can be made arbitrarily small since t_n is also Cauchy. Hence $\mathbf{y}(t_n)$ converges to some vector $\mathbf{y}_T \in \mathbb{R}^d$. It s_n is some other sequence in $(0,T)$ which converges to T, we have:

$$||\mathbf{y}(s_n) - \mathbf{y}(t_n)|| \leq C||\mathbf{y}(0)||e^{CT}|s_n - t_n| \to 0$$

which shows that the limit $\mathbf{y}_T$ is independent of the sequence we choose. Thus we have:

$$\mathbf{y}(T-0) = \mathbf{y}_T, \quad \mathbf{y}'(T-0) = \mathbf{f}(T,\mathbf{y}_T).$$

Now consider the initial value problem $\tilde{\mathbf{y}}'(t) = \mathbf{f}(t,\tilde{\mathbf{y}}(t))$, where $\tilde{\mathbf{y}}(T) = \mathbf{y}_T$. The same local existence Theorem 5.3 (where we put $t_0 = T$ and $\mathbf{y}_0 = \mathbf{y}_T$) allows us to construct a solution on an interval $(T - \delta_2, T + \delta_2)$ and we have:

$$\tilde{\mathbf{y}}'(T+0) = \tilde{\mathbf{y}}'(T) = \mathbf{f}(T,\mathbf{y}_T).$$

Now we can define a function $\mathbf{z} : (-T_1, T_2 + \delta_2)$ where $\mathbf{z}(t) = \mathbf{y}(t)$ on $(-T_1, T_2)$ and $\mathbf{z}(t) = \tilde{\mathbf{y}}(t)$ on $[T_2, T_2 + \delta_2)$. We observe that $\mathbf{z}$ is continuously differentiable and solves the ODE. Thus T_2 can be made larger, which provides a contradiction.

Concerning uniqueness: assume that there exist two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ which both solve the differential equation and $\mathbf{y}_1(0) = \mathbf{y}_2(0) = \mathbf{y}_0$. We already know that they exist for all t. Both of them obey the bound $||\mathbf{y}_j(t)|| \leq ||\mathbf{y}_0||e^{C|t|}$. If $\mathbf{y}_0 = 0$ then both of them are identically zero (thus equal). Hence we may assume that $\mathbf{y}_0 \neq 0$.

Fix some $T > 0$. If $|t| \leq T$, then both vectors $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ will be contained in the closed ball $\overline{B_R(\mathbf{y}_0)}$ with $R := ||\mathbf{y}_0||e^{CT}$.

We know from Lemma 6.3 that there exists some $L < \infty$ such that:

$$||\mathbf{f}(s, \mathbf{x}) - \mathbf{f}(s, \mathbf{z})|| \leq L||\mathbf{x} - \mathbf{z}||, \quad \forall |s| \leq T, \quad \forall \mathbf{x}, \mathbf{z} \in \overline{B_R(\mathbf{y}_0)}.$$

We have the identity:

$$\mathbf{y}_2(t) - \mathbf{y}_1(t) = \int_0^t [\mathbf{f}(s, \mathbf{y}_2(s)) - \mathbf{f}(s, \mathbf{y}_1(s))]ds, \quad \forall |t| \leq T.$$

Let $h(t) := ||\mathbf{y}_2(t) - \mathbf{y}_1(t)||$, with $h(0) = 0$. Assume that $t > 0$. Reasoning as before, we can write:

$$0 \leq h(t) \leq \int_0^t ||\mathbf{f}(s, \mathbf{y}_2(s)) - \mathbf{f}(s, \mathbf{y}_1(s))||ds \leq L \int_0^t h(s)ds \leq L^2 \int_0^t ds_1 \int_0^{s_1} ds_2 h(s_2)...$$

In particular, if $m := \max_{|s| \leq T} h(s)$ we can write:

$$0 \leq h(t) \leq L^N \int_0^t ds_1 \int_0^{s_1} ds_2 ... \int_0^{s_{N-1}} ds_N h(s_N) \leq \frac{mL^N T^N}{N!}, \quad 0 \leq t \leq T.$$

Now the right hand side converges to zero with N, which proves that $h(t)$ is identically zero on $[0, T]$. In a similar way, we can prove that h is zero on $[-T, T]$, hence $\mathbf{y}_1$ and $\mathbf{y}_2$ coincide on that interval. Since T was arbitrary, the two solutions are equal everywhere.

Exercise 2. Show that if $\mathbf{f}(t, \mathbf{x}) = A\mathbf{x}$ where A is an arbitrary $d \times d$ real matrix, its corresponding ODE has a global in time solution.

Exercise 3. Consider the equation

$$y'(t) = \frac{y^2(t)}{1 - y^2(t)}, \quad y(0) = 1/2.$$

1. Define $g : (-1, 1) \mapsto \mathbb{R}$, $g(x) = \frac{x^2}{1-x^2}$. Let $\mathbf{f} : \mathbb{R} \times (-1, 1) \mapsto \mathbb{R}$, $\mathbf{f}(t, x) := g(x)$. Show that $y'(t) = \mathbf{f}(t, y(t))$ and identify d, t_0, I and U.
2. Show that $\mathbf{f} \in C^1(\mathbb{R} \times U)$ and it obeys a local Lipschitz condition.
3. Show that for t near 0 we can rewrite the equation as:

$$[y(t) + 1/y(t) + t]' = 0.$$

4. Show that $y(t) + 1/y(t) = 5/2 - t$ for t near 0. Find $y(t)$.
5. Can y be extended to $t \geq 1/2$?

Exercise 4. Let $\mathbf{f} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ with $\mathbf{f} \in C^1(\mathbb{R}^{d+1})$. Assume that $\mathbf{f}$ obeys a global Lipschitz condition, i.e. there exists a constant $C > 0$ such that

$$||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| \leq C||\mathbf{x} - \mathbf{y}||, \quad \forall t \in \mathbb{R}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Consider the equation $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$, where $\mathbf{y}(0) = \mathbf{y}_0$. Show that there exists a unique solution $\mathbf{y} : \mathbb{R} \mapsto \mathbb{R}^d$ which solves the equation for all $t \in \mathbb{R}$.

Hint. The original differential equation is equivalent with the integral equation:

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_0^t \mathbf{f}(s, \mathbf{y}(s))ds.$$

As in Exercise 1, we assume that a solution only exists on an interval of the form $(-T_1, T_2)$ with $T := T_2 = \min\{T_1, T_2\} < \infty$.

Define $h(t) := ||\mathbf{y}(t) - \mathbf{y}_0||$. We have:

$$\mathbf{y}(t) - \mathbf{y}_0 = \int_0^t \mathbf{f}(s, \mathbf{y}_0)ds + \int_0^t [\mathbf{f}(s, \mathbf{y}(s)) - \mathbf{f}(s, \mathbf{y}_0)]ds.$$

After taking the norms and using the Lipschitz constant (take $t > 0$):

$$h(t) \le \int_0^t ||\mathbf{f}(s, \mathbf{y}_0)||ds + C \int_0^t h(s)ds.$$

If $0 \le t < T$ we have:

$$h(t) \le \int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds + C \int_0^t h(s)ds \le ... \le \left(\int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds \right) e^{CT}, \quad 0 \le t < T.$$

This shows that:

$$||\mathbf{y}(t) - \mathbf{y}_0|| \le \left(\int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds \right) e^{CT}, \quad 0 \le t < T.$$

In other words, the solution always remains inside a closed ball with center at $\mathbf{y}_0$ and radius $R = \left(\int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds \right) e^{CT}$. Let

$$M := \sup_{0 \le s \le T} \sup_{\mathbf{x} \in \overline{B_R(\mathbf{y}_0)}} ||\mathbf{f}(s, \mathbf{x})|| < \infty.$$

Choose some sequence $t_n \in (0, T)$ which converges to T. We have:

$$\mathbf{y}(t_p) - \mathbf{y}(t_q) = \int_{t_q}^{t_p} \mathbf{f}(s, \mathbf{y}(s)), \qquad ||\mathbf{y}(t_p) - \mathbf{y}(t_q)|| \le M|t_p - t_q|.$$

This shows that the sequence $\{\mathbf{y}(t_n)\}_{n \ge 1}$ is Cauchy and converges to some $\mathbf{y}_T$. If $s_n \in (0, T)$ is another sequence which converges to T, we have:

$$\mathbf{y}(t_n) - \mathbf{y}(s_n) = \int_{s_n}^{t_n} \mathbf{f}(s, \mathbf{y}(s)), \qquad ||\mathbf{y}(t_n) - \mathbf{y}(s_n)|| \le M|t_n - s_n|$$

which proves that $\mathbf{y}_T$ is independent of the sequence we choose, hence $\mathbf{y}(T - 0)$ exists and equals $\mathbf{y}_T$, and:

$$\mathbf{y}'(T - 0) = \mathbf{f}(T, \mathbf{y}_T).$$

Reasoning as in Exercise 1, we can locally extend $\mathbf{y}$ to the interval $[T, T + \delta)$, thus contradicting the maximality of the interval $(-T_1, T)$.

Now let us prove uniqueness. Assume that both $\mathbf{y}_1$ and $\mathbf{y}_2$ solve the integral equation. Define $h(t) := ||\mathbf{y}_1(t) - \mathbf{y}_2(t)||$. We have:

$$\mathbf{y}_1(t) - \mathbf{y}_2(t) = \int_0^t [\mathbf{f}(s, \mathbf{y}_1(s)) - \mathbf{f}(s, \mathbf{y}_2(s))]ds, \quad 0 \le h(t) \le C \int_0^t h(s)ds.$$

As in Exercise 1 we can show that $h(t) = 0$ for all t and we are done.

Exercise 5. Let $\mathbf{g} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by $\mathbf{g}(\mathbf{x}) = [-x_2, x_1]$ and consider the equation

$$\mathbf{y}'(t) = \mathbf{g}(\mathbf{y}(t)), \quad \mathbf{y}(0) = [1, 0].$$

1. Show that $y_1'(t) = -y_2(t)$ and $y_2'(t) = y_1(t)$, with $y_1(0) = 1$ and $y_2(0) = 0$.
2. Prove that the solution is unique and global and time, and moreover, $y_1^2(t) + y_2^2(t) = 1$ for all t.
3. Use uniqueness in order to show that $y_1(t) = y_1^2(t/2) - y_2^2(t/2)$ and $y_2(t) = 2y_1(t/2)y_2(t/2)$.
4. Prove that there must exist a $T > 0$ such that $y_1(T) = 1$ and $y_2(T) = 0$.
5. Use uniqueness in order to show that $y_1(t) = y_1(t + T)$ and $y_2(t) = y_2(t + T)$ for all t.
6. Denote by $2P$ the smallest positive T in (5). Show that $y_1(P) = -1$ and $y_2(P) = 0$.
7. Use uniqueness and (6) to show that $y_1(t) = -y_1(P - t)$ and $y_2(t) = y_2(P - t)$.
8. Use uniqueness and (6) to show that $y_1(t) = -y_1(P + t)$ and $y_2(t) = -y_2(P + t)$.
9. Use uniqueness and show that $y_1(t) = y_1(-t)$ and $y_2(t) = -y_2(-t)$.
10. Use (7) and (8) to show that $y_1(P/2) = 0$ and $y_1(3P/2) = 0$.
11. Show that $y_1'(0) = 0$, $y_1''(0) = -1$, $y_1'''(0) = 0$, $y_1^{(4)}(0) = 1$,... . Compute the Taylor series of $y_1(t)$ around 0 and show that it has an infinite radius of convergence. Can you recognize the function? What about the number P?

Hints.

(2). Show that $[y_1^2(t) + y_2^2(t) - 1]' = 0$ for all t.

(3). Define $\tilde{y}_1(t) := y_1^2(t/2) - y_2^2(t/2)$ and $\tilde{y}_2(t) := 2y_1(t/2)y_2(t/2)$. Prove that $\tilde{\mathbf{y}}'(t) = \mathbf{g}(\tilde{\mathbf{y}}(t))$ and $\tilde{\mathbf{y}}(0) = [1, 0]$.

(4). Show that when t is slightly larger than zero, y_1 decreases while y_2 increases and becomes positive. This remains true until y_1 hits zero, and y_2 necessarily equals 1. Then y_1 continues to decrease and becomes more and more negative, making y_2 to decrease until it hits zero. Simultaneously, y_1 must equal -1. After that, y_2 becomes negative which makes y_1 to increase again until it hits zero. At that point, y_2 must be -1. Finally, y_1 will continue to increase as long as y_2 is negative, reaching the value 1, and necessarily, y_2 will be zero.

(6) Let $t = 2P$ in (3). We have $1 = y_1(2P) = y_1^2(P) - y_2^2(P) = 2y_1^2(P) - 1$ and $0 = y_2(2P) = 2y_1(P)y_2(P)$. The first identity implies that $|y_1(P)| = 1$; this implies that $y_2(P) = 0$. Hence $y_1(P)$ equals either $+1$ or -1. But it cannot equal $+1$, because we assumed that the smallest positive value of t for which we come back to the original initial condition was $2P$.