Notes for *Analyse 1* and *Analyse 2*

Horia Cornean, 9/04/2015.

Contents

These notes are strongly inspired by the books *Principles of Mathematical Analysis* by Walter Rudin and *Topology from the Differentiable Viewpoint* by John Milnor. Some of the theorems below can be formulated in a more general setting than the one of metric spaces, but the metric space structure brings important simplifications and clarity. Fundamental results like the Brouwer, Schauder and Kakutani Fixed Point Theorems, the Hairy Ball Theorem, the Tietze Extension Theorem, and the Jordan Curve Theorem are not in the curriculum, but this does not make them less important. All proofs are quite detailed and self-contained, and are at the level of hard-working second year undergraduate students.

In the first chapter we construct the field of real numbers as decimals in the spirit of Otto Stolz (1885). Several comments on the actual construction can be found at the beginning of the chapter.

The next two chapters deal with point set topology in metric spaces. In particular we prove the equivalence between compact and sequentially compact sets in general metric spaces, and the Bolzano-Weierstrass and Heine-Borel Theorems in Euclidean spaces.

Chapter four deals with continuous functions on metric spaces. We show the equivalence between continuity, sequential continuity, and 'returning open sets into open sets'. We show that a continuous function defined on a compact set is uniformly continuous.

Chapter five proves the Banach Fixed Point Theorem. Chapter six is based on the previous one and investigates the local existence and uniqueness of solutions to first order differential equations.

Chapter seven contains the Implicit Function Theorem. Its proof is based on Banach's Fixed Point Theorem. Chapter eight deals with the Inverse Function Theorem, whose proof is shown to be a consequence of the Implicit Function Theorem.

Chapter nine contains the proof of the Brouwer Fixed Point Theorem. We follow the strategy of C.A. Rogers from the paper *A Less Strange Version of Milnor's Proof of Brouwer's Fixed Point Theorem*, appeared in Amer. Math. Monthly. **87** 525-527 (1980). We give many more details and the presentation is completely analytic and self-contained, based on the previous six chapters. We also prove that any convex body is homeomorphic with the closed unit ball.

Chapter ten contains the Schauder Fixed Point Theorem, presented as a consequence of Brouwer's Fixed Point Theorem.

Chapter eleven presents the Kakutani Fixed Point Theorem. Its proof is an adaptation of that of S. Kakutani in *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal **8**(3), 457-459 (1941). This theorem is another consequence of Brouwer's Fixed Point Theorem.

Chapter twelve contains the proof of the existence of a Nash equilibrium for a finite game with two players. The original paper of J. Nash is *Non-cooperative games*, Annals of Math. **54** (2), 286-295 (1951). Our proof is based on the ideas of J. Geanakoplos in *Nash and Walras equilibrium via Brouwer*, Economic Theory **21**, 585-603 (2003).

Chapter thirteen contains an analytic proof of the Hairy Ball Theorem, and follows the strategy used by J. Milnor in the paper *Analytic proofs of the hairy ball theorem and the Brouwer fixed point theorem*, appeared in Amer. Math. Monthly **85**, 521-524 (1978).

Chapter fourteen contains the Jordan Curve Theorem and is inspired by a paper of R. Maehara entitled *The Jordan curve theorem via the Brouwer fixed point theorem*, which appeared in Amer. Math. Monthly **91**(10), 641-643 (1984). We give many more details and in particular, we prove a simple version of the Tietze Extension Theorem in $\mathbb{R}^2$ based on an extension formula due to Hausdorff.

1 On the construction of real numbers as decimals

This chapter is inspired by Martin Klazar's manuscript entitled *Real numbers as infinite decimals and irrationality of* $\sqrt{2}$, available at http://arxiv.org/abs/0910.5870. Note that the main ideas can be traced back at least to Otto Stolz in *Vorlesungen über Allgemeine Arithmetik. Erster Theil: Allgemeines und Arithmetik der Reelen Zahlen*, published in 1885.

The construction we give here is elementary but it might be difficult to follow without a road-map. Here is a short description of what we do:

- We assume that the order relation and the properties of operations involving the natural numbers $\mathbb{N}$ and integers $\mathbb{Z}$ are known.

- We define the set of decimals $\mathbb{D}$ together with the natural lexicographic order relation on it. We use the decimal representation from pedagogical reasons. The ordered set of real numbers $\mathbb{R}$ is introduced in Definition 1.1. Proposition 1.2 shows that the terminating decimals are dense in $\mathbb{R}$.

- We introduce the notions of supremum and infimum, and we prove in Theorem 1.6 that any bounded set of real numbers has a supremum and an infimum. Thus in this construction, the completeness axiom is a theorem.

- In paragraph 1.1.3 we identify $\mathbb{N}$ and $\mathbb{Z}$ with a certain class of terminating decimals and we define their addition and multiplication. Moreover, we implement the multiplication and division with powers of 10 by right/left shifts of the comma.

- Paragraph 1.2 studies sequences of terminating decimals, which will be later used in paragraph 1.3 in order to define addition and multiplication of arbitrary real numbers. The main ingredients are the Cauchy property, strong and formal convergence.

- In paragraph 1.4 we define addition and multiplication of arbitrary real numbers; the idea is the following: given two real numbers, we approximate them with sequences of terminating decimals which we know how to add and multiply from paragraph 1.2, then we show that the result we get converges to some unique real number. Commutativity of addition and multiplication, distributivity and compatibility with the order relation are also proved there.

- The last ingredient is to show that every non-zero real number has a multiplicative inverse. We first show in (1.9) that we can compute the inverse of any integer. Then we can define the set of rational numbers, and in Theorem 1.28 we prove that a number is rational if and only if it corresponds to an eventually periodic decimal. In the very end, we construct a multiplicative inverse for any non-zero real number through a limiting procedure using rational approximants.

1.1 The set of real numbers

Let $\mathbb{D}$ denote the set of all semi-infinite sequences (called decimals from now on) of the type

$$d = \pm d_n d_{n-1} ... d_0, d_{-1} d_{-2} ...$$

where $n \geq 0$ is finite, each $d_j \in \{0, 1, 2, ..., 9\}$ and $d_n > 0$ if $n > 0$. By convention, if $j > n$ we put $d_j = 0$.

The zero element in $\mathbb{D}$ is also denoted by 0 and corresponds to choosing all coefficients equal to zero. All non-zero decimals with a '+' sign are called positive (from now on we will omit writing '+' in front of a positive decimal), while those with '−' are called negative. By convention, if d is negative, then $-d$ is positive (i.e. the minus sign of d is erased).

Let $\mathbb{T} \subset \mathbb{D}$ be the set of all (terminating) decimals which have the property that there exists some integer $J \in \mathbb{Z}$ such that $d_j = 0$ for all $j \leq J$. In other words, if $d \in \mathbb{T}$ then only a finite number of coefficients d_j are different from zero. In order to simplify notation, we shall omit writing the infinitely many zeros of such a decimal. For example, instead of writing $1,10000...$ we only write $1,1$. We also denote by $\widetilde{\mathbb{T}}$ the set of decimals for which there exists some $J \in \mathbb{Z}$ such that $d_j = 9$ for all $j \leq J$.

1.1.1 An order relation on $\mathbb{D}$

We introduce a transitive order relation $<<$ on D (called 'less than') in the following way. If d is positive, then $0 << d$. If d is negative, then $d << 0$. Due to the transitivity property we want to impose, any negative decimal must be 'less than' any positive decimal.

If d and t are two positive decimals, then by definition, we have that $d << t$ if there exists some $J \in \mathbb{Z}$ such that $d_J < t_J$ while $d_j = t_j$ for $j > J$. For example, if $d = 34,555546$ and $t = 117,00399999...$ then $d << t$; here $J = 2$ and $d_2 = 0 < 1 = t_2$. Another important example: if $d = 0,9999999...$ and $t = 1$ then $d << t$; here $J = 0$ and $d_0 = 0 < t_0 = 1$.

The second example is particularly important. We see that we *cannot* find any decimal $e \in \mathbb{D}$ such that:

$$0,9999... << e << 1.$$

For any terminating decimal in $t \in \mathbb{T}$ where the last non-zero coefficient is $t_J \in \{1, ..., 9\}$, we can find an element $d \in \widetilde{\mathbb{T}}$ such that $d_j = t_j$ for $j > J$, $d_J = t_J - 1$ and $d_j = 9$ for all $j < J$. Let us call such a pair a **jump**. The two elements of a given jump will be identified as one element.

A decimal **does not** belong to a jump if and only if it contains infinitely many coefficients which obey $1 \leq d_j \leq 8$.

Concerning negative decimals: if both d and t are negative, then we say that $d << t$ if we have $-t << -d$.

Definition 1.1. *The set of real numbers $\mathbb{R}$ is defined to be the collection containing the decimal 0 and all the elements of $\mathbb{D}$ which are neither in $\mathbb{T}$ nor in $\widetilde{\mathbb{T}}$, together with all the possible jumps.*

In order to make clear that we work with real numbers and not just decimals, we shall use the notation $[d]$ which means just d if the decimal is not an element of a jump. For example, if $d = 0,1111...$ then $[d] = d$; at the same time,

$$[0,099999....] = [0,1000...].$$

The order relation induced by $<<$ in $\mathbb{R}$ is simply denoted by $<$. Thus if we write $[d] < [t]$ it means that $d << t$ and d, t do not belong to the same jump. Moreover, given any two different real numbers $[d] \neq [t]$ we either have $[d] < [t]$ or $[t] < [d]$. Now we can prove the density of terminating decimals:

Proposition 1.2. *Given any two real numbers $[d] < [t]$, we can find a terminating decimal $e \in \mathbb{T}$ such that $[d] < [e] < [t]$.*

Proof. If $[d]$ is negative and $[t]$ positive, then we can take $[e] = 0$. Now assume that $0 < [d] < [t]$. Two alternatives are possible:

- $[d]$ is a jump and we can choose d to be a terminating decimal. We also have $d << t$ because t and d are not in the same jump. Thus there exists some $J \in \mathbb{Z}$ such that $d_j = t_j$ for $j > J$ and $d_J < t_J$. Because $d \in \mathbb{T}$, there exists some $n \leq J$ such that $d_j = 0$ if $j \leq n$. Now choose $e \in \mathbb{T}$ to be identical with d up to the coefficient with index n, then put $e_{n-1} := 1$ and $e_j := 0$ for $j < n - 1$. We have $e \in \mathbb{T}$, $e << t$ and $[d] < [e] < [t]$.

- $[d]$ is not a jump, i.e. it contains infinitely many coefficients which obey $1 \leq d_j \leq 8$ for $j < J$. Then define $e \in \mathbb{T}$ to be identical with d up to the coefficient J, then turn the first $1 \leq d_j \leq 8$ into a 9, then set all its following coefficients to be zero. Again we have $e \in \mathbb{T}$, $e << t$ and $[d] < [e] < [t]$.

Finally, if $[d] < [t] < 0$ then we have $0 < [-t] < [-d]$, hence there exists $[e] > 0$ with $e \in \mathbb{T}$ such that $[-t] < [e] < [-d]$. By changing signs we have $[d] < [-e] < [t]$ and the proof is over. □

1.1.2 Supremum and infimum

Until now we only defined an order relation on $\mathbb{R}$. We say that $[d] \leq [t]$ if either $[d] = [t]$ or $[d] < [t]$. This order relation is total.

We say that a set $S \subset \mathbb{R}$ is *bounded from above* if there exists some $[M] \in \mathbb{R}$ such that $[x] \leq [M]$ for all $[x] \in S$. A set $S \subset \mathbb{R}$ is *bounded from below* if there exists some $[m] \in \mathbb{R}$ such that $[x] \geq [m]$ for all $[x] \in S$.

Definition 1.3. *We say that $[a] \in \mathbb{R}$ is a* **supremum** *of a set $S \subset \mathbb{R}$ if two properties hold true:*

1. *$[a]$ is an* upper bound *for S, which means that $[x] \leq [a]$ for all $[x] \in S$;*
2. *$[a]$ is the smallest possible upper bound, i.e. no $[b] < [a]$ can be an upper bound; in other words, given $[b] < [a]$, we can find some element $[x_b] \in S$ such that $[b] < [x_b] \leq [a]$.*

If it exists, the supremum of a set S is unique; let us prove it. Assume that there exist two different real numbers $[\alpha]$ and $[\beta]$ which both obey the above two conditions. Assume that $[\alpha] < [\beta]$. Then the second property applied for $[\beta]$ provides us with an element $[x] \in S$ such that $[\alpha] < [x] \leq [\beta]$. But this contradicts the fact that $[\alpha]$ is an upper bound for S. In a similar way, one can prove that $[\beta] < [\alpha]$ also leads to a contradiction. Hence they cannot be different.

The supremum is denoted by $\sup(S)$.

Definition 1.4. *We say that $[a] \in \mathbb{R}$ is an* **infimum** *of a set $S \subset \mathbb{R}$ if two properties hold true:*

1. *$[a]$ is a* lower bound *for S, which means that $[x] \geq [a]$ for all $[x] \in S$;*
2. *$[a]$ is the largest possible lower bound, i.e. no $[b] > [a]$ can be a lower bound; in other words, given $[b] > [a]$, we can find some element $[x_b] \in S$ such that $[a] \leq [x_b] < [b]$.*

If it exists, the infimum of a set S is unique (the proof is similar as for the supremum) and denoted by $\inf(S)$.

Lemma 1.5. *If $S \subset \mathbb{R}$, we denote by $-S$ the set $\{[x] \in \mathbb{R} : \ [-x] \in S\}$. Then if S has a supremum, the set $-S$ has an infimum and $\inf(-S) = -\sup(S)$; moreover, if S has an infimum, the set $-S$ has a supremum and $\sup(-S) = -\inf(S)$.*

Proof. We only prove the first part, the second one being analogous. If S has a supremum, let $[a] := -\sup(S)$. Since $[x] \leq \sup(S)$ for all $[x] \in S$, we have $[a] \leq [-x]$ for all $[x]$, hence $[a]$ is a lower bound for $-S$.

Now let us show that $[a]$ is the largest lower bound. Take some $[b]$ larger than $[a]$. Then $[-b] < [-a] = \sup(S)$. Thus we may find some $[x_b] \in S$ such that $[-b] < [x_b] \leq -[a]$. This implies $[a] \leq [-x_b] < [b]$ with $[-x_b] \in -S$ which shows that $[b]$ cannot be a lower bound for $-S$ and we are done. □

Theorem 1.6. *Let $S \subset \mathbb{R}$ be nonempty and bounded from above (below). Then the supremum (infimum) of S exists.*

Proof. We first consider the case when S only contains positive numbers, the general case being treated at the end of the proof.

Since S is bounded from above, there exists a decimal M such that $x << M$ for every decimal x such that $[x] \in S$. We can choose $M = M_n 000...0, 000...$ with a finite n. Thus given any decimal x with $[x] \in S$ we must have $x_j = 0$ if $j > n$, i.e. every such decimal has at most n nonzero coefficients with positive index. Therefore, all decimals corresponding to the elements of S have the form

$$x = x_n x_{n-1} ... x_0, x_{-1} x_{-2} ...$$

where the coefficient x_n is allowed to be zero. Now define a_n to be the largest possible x_n when $[x]$ takes all possible values in S. Note that a_n may be zero.

Define the set

$$S_1 := \{[x] \in S : x_n = a_n\} \subset S.$$

Define a_{n-1} to be the largest possible component x_{n-1} when $[x] \in S_1$. In general, if $k \geq 2$ we have:

$$S_k := \{[x] \in S : x_n = a_n, x_{n-1} = a_{n-1}, ..., x_{n-k+1} = a_{n-k+1}\} \subset S_{k-1} \subset ... \subset S,$$

and then we define a_{n-k} to be the largest possible x_{n-k} when $[x]$ takes all possible values in S_k. In this way we define the decimal $a = a_n a_{n-1} ... a_0, a_{-1}....$

Let us show that $[a]$ is an upper bound for S. Take any x with $[x] \in S$. If $x_n < a_n$ then $x << a$ and $[x] \leq [a]$. If $x_n = a_n$ then $[x] \in S_1$, thus $x_{n-1} \leq a_{n-1}$. If we have a strict inequality then we are done, otherwise we continue. In this way we show that $x_j \leq a_j$ for every $j \leq n$, hence $[x] \leq [a]$.

Let us show that $[a]$ is the smallest upper bound. Take any $[b] \in \mathbb{R}$ with $[b] < [a]$. From Proposition 1.2 we know that there exists a terminating decimal $e = e_n e_{n-1} ... e_0, e_{-1} ... e_{-J} \in \mathbb{T}$ such that $[b] < [e] < [a]$. Here J is finite. Now choosing k large enough (compared to J), we have that any decimal x with $[x] \in S_k$ will have its first k coefficients equal to the first k coefficients of a, hence $[e] < [x]$ and we are done.

In a similar way we can construct the infimum of S when S only contains positive numbers.

Now let us extend the result to the case in which S only contains negative numbers. Define the set $-S := \{[-x] : [x] \in S\}$; it will only contain positive numbers. If S is bounded from below, then $-S$ is bounded from above and Lemma 1.5 implies that $\inf(S) = -\sup(-S)$. Clearly, S is bounded from above by zero and $\sup(S) = -\inf(-S)$.

If S contains both positive and negative numbers, we can write $S = S_+ \cup S_-$ where S_+ includes the positive elements and eventually 0, while S_- includes the negative numbers. Then $\sup(S) = \sup(S_+)$ and $\inf(S) = \inf(S_-)$.

□

As an example, let us consider the set $S = (0, 1)$, i.e. the open interval containing all real numbers such that $0 < [x] < [1]$. The above construction gives us $a = 0,9999...$, i.e. $\sup(S) = [1]$. Note that the supremum of a set is not necessarily an element of it. If it belongs to it, then it is called maximum.

1.1.3 Adding and multiplying terminating decimals

Any terminating decimal of the type $\pm x_n x_{n-1} ... x_0, 000...$ can be uniquely identified with the integer

$$\pm(10^n x_n + 10^{n-1} x_{n-1} + ... + 10 x_1 + x_0) \in \mathbb{Z}.$$

Thus we can define the sum and the product of any two terminating integers of the above type through the identification with integers. Note that the effect of multiplying with 10^k with $k \geq 1$ is a shift of the comma with k places to the right, like for example:

$$10^2 \cdot x_n x_{n-1} ... x_0, 000... = x_n x_{n-1} ... x_0 00, 000.$$

By definition, we denote the terminating decimal $0,1000...$ with 10^{-1}, the decimal $0,01000...$ with 10^{-2} and so on. By convention, when we multiply a terminating decimal with 10^{-k}, the comma is shifted k places to the left.

Given two decimals $x = x_N x_{N-1} ... x_0, x_{-1} ... x_{-M} 000...$ and $y = y_P y_{P-1} ... x_0, y_{-1} ... y_{-Q} 000...$ with $Q \geq M$ then we have:

$$x = 10^{-Q} \cdot x_N x_{N-1} ... x_0 x_{-1} ... x_{-M} 0...00, 00..., \quad y = 10^{-Q} \cdot y_P y_{P-1} ... x_0 y_{-1} ... y_{-Q}, 000$$

hence:

$$x + y := 10^{-Q} \cdot (x_N x_{N-1} ... x_0 x_{-1} ... x_{-M} 0...00 + y_P y_{P-1} ... x_0 y_{-1} ... y_{-Q})$$

and

$$x \cdot y := 10^{-2Q} \cdot (x_N x_{N-1} ... x_0 x_{-1} ... x_{-M} 0...00 \cdot y_P y_{P-1} ... x_0 y_{-1} ... y_{-Q}),$$

according to the previously introduced rules. Also, when we multiply two negative numbers we get a positive one, and when we multiply a negative terminating decimal with a positive one, the result is negative.

Until now we have defined how to add, subtract and multiply any two terminating decimals, hence to real numbers which are jumps. In the next sections we shall extend these operations to all real numbers, and define division. **From now on, we will not make a notational distinction between a terminating decimal and its corresponding real number (jump).**

1.2 Strong and formal convergence of sequences in $\mathbb{T}$

The absolute value of any decimal is defined in the natural way (by "throwing away the sign"):

$$0 \leq | \pm x_n x_{n-1} ... x_0, x_{-1} ... x_{-m} ...| := [x_n x_{n-1} ... x_0, x_{-1} ... x_{-m} ...] \in \mathbb{R}.$$

For terminating decimals we also have the triangle inequality:

$$|x + y| \leq |x| + |y|, \quad \forall x, y \in \mathbb{T},$$

and

$$|x \cdot y| = |x| \cdot |y|,$$

properties which can be easily inferred using the properties of $\mathbb{Z}$. Given any decimal $x = \pm x_N x_{N-1} ... x_0, x_{-1} x_{-2} ...$ not necessarily terminating, we define its n'th order truncation:

$$T_n(x) := \pm x_N x_{N-1} ... x_0, x_{-1} x_{-2} ... x_{-n} 000... \in \mathbb{T}, \quad n \geq 1.$$

We have the following very useful representation of jumps. Given two decimals x and y which belong to the same jump, then we must have:

$$\forall k \geq 1, \ \exists N_k \geq 1 : \ |T_n(x) - T_n(y)| \leq 10^{-k}, \ \forall n \geq N_k. \tag{1.1}$$

As an example, let $x = 1,000...$ and $y = 0,9999....$ Then we have $T_1(x) - T_1(y) = 10^{-1}$, $T_2(x) - T_2(y) = 10^{-2}$, ..., $T_n(x) - T_n(y) = 10^{-n}$. Thus given any k, by choosing for example $N_k := k$, then for every $n \geq N_k$ we have $10^{-n} \leq 10^{-k}$.

The reversed implication is also true, but this fact is more complicated; let us prove it now. Assume that (1.1) holds true, but x and y do not belong to the same jump. Let us assume that $0 < [x] < [y]$. From Proposition 1.2 we obtain some $e \in \mathbb{T}$ such that $x << e$ and $e << y$. Assume that $e = e_N e_{N-1} ... e_0, e_{-1} ... e_{-M}$. Then we must have

$$10^{-M-1} << e - T_n(x), \quad 10^{-M-1} << T_n(y) - e, \quad \forall n > M.$$

Thus:

$$10^{-M-2} << 2 \cdot 10^{-M-1} << T_n(y) - T_n(x), \quad \forall n > M,$$

hence the right hand side cannot be made arbitrarily small as stated in (1.1). In other words, we have just proved that if x and y are not in the same jump, then:

$$\exists M \in \mathbb{N} : \ 10^{-M-2} \leq |T_n(x) - T_n(y)|, \quad \forall n \geq M. \tag{1.2}$$

1.2.1 Strong convergence

We denote by $\{x(n)\}_{n \geq 1} \subset \mathbb{T}$ any sequence consisting of terminating decimals, i.e. $(x(n))_{-j} = 0$ if j is large enough.

Definition 1.7. *We say that such a sequence has a strong limit* $[x] \in \mathbb{R}$ *if:*

$$\forall k \in \mathbb{N},\ \exists N_k \geq 1 : \ |x(n) - T_n(x)| \leq 10^{-k},\ \forall n \geq N_k. \tag{1.3}$$

In this case we write:

$$\text{slim}_{n\to\infty} x(n) = [x].$$

There is an apparent ambiguity in this definition, because $[x]$ might be a jump, hence there are two possibilities for choosing x in (1.3). But the strong limit $[x]$ seen as a real number is unique if it exists; let us prove this. Assume that (1.3) hold true both for $[x]$ and $[y]$. Then using the triangle inequality we have:

$$|T_n(x) - T_n(y)| \leq |T_n(x) - x(n)| + |x(n) - T_n(y)|,$$

where the right hand side can be made arbitrarily small when n is large enough. Thus according to (1.1), x and y must be in the same jump, hence they both define the same real number.

As an example, let us consider the sequence $x(1) = 0,9$, $x(2) = 0,99$, $x(3) = 0,999$, ... Then the strong limit of this sequence exists and equals [1].

Lemma 1.8. *Let* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *be such that* $\text{slim}_{n\to\infty} x(n) = [x]$*. Then the sequence is bounded, i.e. there exists some* $J \in \mathbb{N}$ *such that* $|x(n)| \leq 10^J$ *for all* $n \geq 1$*.*

Proof. We can find some J_1 such that $|x| \leq 10^{J_1}$, hence $|T_n(x)| \leq 10^{J_1}$ for every n. Choose $k = 0$ in (1.3). This implies the existence of some N_0 such that $|x(n) - T_n(x)| \leq 1$ if $n \geq N_0$. The triangle inequality leads to:

$$|x(n)| \leq |T_n(x)| + 1 \leq 10^{J_1} + 1 \leq 10^{J_1+1}, \quad \forall n \geq N_0.$$

Thus we have:

$$|x(n)| \leq \max\{|x(1)|, ..., |x(N_0)|, 10^{J_1+1}\}, \quad \forall n \geq 1,$$

where the right hand side can be bounded from above by some positive power of 10. □

Lemma 1.9. *Let* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *and* $\{y(n)\}_{n\geq 1} \subset \mathbb{T}$ *be such that* $\text{slim}_{n\to\infty} x(n) = 0$ *and* $\text{slim}_{n\to\infty} y(n) = [y]$*. Then* $z(n) := x(n) + y(n)$ *is also strongly convergent and* $\text{slim}_{n\to\infty} z(n) = [y]$*.*

Proof. We have:

$$|z(n) - T_n(y)| \leq |x(n)| + |y(n) - T_n(y)|, \quad n \geq 1.$$

Fix $k \in \mathbb{N}$. There exists $N_k^{(1)}$ such that $|x(n)| \leq 10^{-k-1}$ for all $n \geq N_k^{(1)}$. Also, there exists $N_k^{(2)}$ such that $|y(n) - T_n(y)| \leq 10^{-k-1}$ for all $n \geq N_k^{(2)}$. Hence if $n \geq \max\{N_k^{(1)}, N_k^{(2)}\}$ we have:

$$|z(n) - T_n(y)| \leq 2 \cdot 10^{-k-1} \leq 10^{-k}$$

and we are done.

□

Lemma 1.10. *Let* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *be bounded and* $\{y(n)\}_{n\geq 1} \subset \mathbb{T}$ *be such that* $\text{slim}_{n\to\infty} y(n) = 0$*. Then* $z(n) := x(n) \cdot y(n)$ *is also strongly convergent and* $\text{slim}_{n\to\infty} z(n) = 0$*.*

Proof. Because $\{x(n)\}_{n\geq 1}$ is bounded, there exists $J \in \mathbb{N}$ such that $|x(n)| \leq 10^J$. Then:

$$|z(n)| \leq |x(n)| \cdot |y(n)| \leq 10^J \cdot |y(n)|, \quad n \geq 1.$$

Fix $k \in \mathbb{N}$. There exists N_k' such that $|y(n)| \leq 10^{-k-J}$ for all $n \geq N_k'$. Hence if $n \geq N_k'$ we have:

$$|z(n)| \leq 10^{-k}$$

and we are done.

□

Lemma 1.11. *Assume that $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ converges strongly to $[x]$ and $\{y(n)\}_{n\geq 1} \subset \mathbb{T}$ converges strongly to $[y]$. If $y(n) \leq x(n)$ for all n, then $[y] \leq [x]$.*

Proof. Assume that the conclusion is not true, i.e. $[x] < [y]$. According to (1.2), there must exist some M such that

$$T_n(x) - T_n(y) \leq -10^{-M-2}, \quad \forall n \geq M.$$

We have:

$$\begin{aligned} 0 \leq x(n) - y(n) &= (x(n) - T_n(x)) + T_n(x) - T_n(y) + (T_n(y) - y(n)) \\ &\leq -10^{-M-2} + (x(n) - T_n(x)) + (T_n(y) - y(n)), \quad \forall n \geq M. \end{aligned}$$

Due to (1.3), by choosing n large enough we can make both $x(n) - T_n(x)$ and $T_n(y) - y(n)$ less than 10^{-M-3}, hence we arrive at a contradiction.

□

1.2.2 Formal convergence

Definition 1.12. *We say that $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ has a formal limit $x \in \mathbb{D}$ if:*

$$\forall j \in \mathbb{N},\ \exists M_j \geq 1 : \ T_j(x(n)) = T_j(x), \quad \forall n \geq M_j. \tag{1.4}$$

In this case we write:

$$\mathrm{flim}_{n\to\infty} x(n) = x.$$

The formal convergence tells us that given some j and if n becomes larger than some critical value M_j, the decimal $x(n)$ will coincide with x at least up to the coefficient of index $-j$. It is easy to prove that the formal limit is unique if it exists.

Lemma 1.13. *Assume that $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ has a formal limit $x \in \mathbb{D}$. Then the sequence is also strongly convergent, with the strong limit $[x] \in \mathbb{R}$.*

Proof. If $n, j \in \mathbb{N}$ we have:

$$x(n) - T_n(x) = x(n) - T_j(x(n)) + T_j(x(n)) - T_j(x) + T_j(x) - T_n(x).$$

If we impose that $n \geq j$, then we see that both decimals $x(n) - T_j(x(n))$ and $T_j(x) - T_n(x)$ will have zero coefficients at least up to the index $-j$. Hence:

$$|x(n) - T_n(x)| \leq 2 \cdot 10^{-j} + |T_j(x(n)) - T_j(x)|, \quad n \geq j.$$

If $n \geq M_j$ as given by (1.4), then:

$$|x(n) - T_n(x)| \leq 2 \cdot 10^{-j}, \quad n \geq M_j.$$

Now it is enough to choose $j = k + 1$ and (1.3) is proved. □

Definition 1.14. *We say that $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ is increasing if $x(n) \leq x(n+1)$ for all $n \geq 1$. The sequence is decreasing if $x(n+1) \leq x(n)$ for all $n \geq 1$. A sequence is called monotone if it is either increasing or decreasing.*

Lemma 1.15. *Assume that $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ is bounded from above (below) and is increasing (decreasing). Then the sequence has a formal limit.*

Proof. Assume that the sequence is increasing and bounded from above by some 10^J. Then the set $S := \{x(n) :\ n \geq 1\}$ is bounded from above, hence according to Theorem 1.6 it has a supremum $[a] \in \mathbb{R}$, where $a = a_J a_{J-1} ... a_0, a_{-1} ....$ The decimal a was constructed in the following way: a_J was the largest possible J'th coefficient among all decimals $x(n)$. In particular, there must exist some n_J such that $x(n_J)$ has a_J as its J'th coefficient. Because the sequence is increasing, $x(n_J) \leq x(n)$ if $n \geq n_J$, hence the J'th coefficient of such an $x(n)$ will remain equal to a_J. Reasoning in the same way, there must exist some $n_{J-1} \geq n_J$ such that the $J-1$'th coefficient of $x(n_{J-1})$ equals a_{J-1}, hence all decimals with $n \geq n_{J-1}$ will have the J'th and $J-1$'th coefficients equal to a_J and a_{J-1}, respectively. By induction, we can prove that given $j \geq 1$ there must exist some M_j such that $T_j(x(M_j)) = T_j(a)$. Since $x(M_j) \leq x(n)$ if $n \geq M_j$ we must have that $T_j(x(n)) = T_j(a)$ and (1.4) follows.

The decreasing case is proved analogously, and the formal limit is related to the infimum of the set S. □

1.2.3 Sequences with the Cauchy property

Definition 1.16. *We say that* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *has the Cauchy property if*

$$\forall k \in \mathbb{N},\ \exists M_k \in \mathbb{N} : \quad |x(m) - x(n)| \leq 10^{-k},\ \forall m, n \geq M_k. \tag{1.5}$$

We first show that strongly convergent sequences also have the Cauchy property.

Proposition 1.17. *Assume that* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *has a strong limit* $[x] \in \mathbb{R}$*. Then the sequence has the Cauchy property.*

Proof. According to (1.3), given $j \geq 1$ we can find N_j such that $|x(n) - T_n(x)| \leq 10^{-j}$ for all $n \geq N_j$. If $m \geq n \geq j$ we have $|T_m(x) - T_n(x)| \leq 10^{-j}$. Using the triangle inequality we have:

$$|x(m)-x(n)| \leq |x(m)-T_m(x)|+|T_m(x)-T_n(x)|+|T_n(x)-x(n)| \leq 3\cdot 10^{-j},\ \forall m \geq n \geq \max\{j, N_j\}.$$

Now choose $j = k+1$ and $M_k := \max\{k+1, N_{k+1}\}$ and we are done. □

In the rest of this subsection we shall show that the reverse of the above lemma also holds true, i.e. any sequence with the Cauchy property is strongly convergent. This fact is more complicated and needs some preparatory results.

Lemma 1.18. *Assume that* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *has the Cauchy property. Then the sequence is bounded, i.e. there exists* $J \in \mathbb{N}$ *such that* $|x(n)| \leq 10^J$ *for all* $n \geq 1$.

Proof. Let $k = 1$ in (1.5). Then $|x(m)-x(M_1)| \leq 10^{-1}$ for all $m \geq N_1$, hence by using the triangle inequality:

$$|x(m)| \leq |x(M_1)| + 10^{-1}, \quad \forall m \geq M_1.$$

Thus

$$|x(n)| \leq \max\{|x(1), |x(2)|, ..., |x(N_1)| + 10^{-1}\}, \quad \forall n \geq 1,$$

and we are done. □

Lemma 1.19. *Assume that* $\{x(n)\}_{n\geq 1} \subset \mathbb{T}$ *is bounded. Then there exists a monotone subsequence* $\{x(n_j)\}_{j\geq 1} \subset \{x(n)\}_{n\geq 1}$.

Proof. We know that the set $S = \{x(n) : \ n \geq 1\}$ is bounded from above, thus it must have a supremum $[a]$. There are two alternatives: either a does not belong to S, or a belongs to S.

We first assume that a does not belong to S. In this case, we put $n_1 := 1$ and we have $x(n_1) < [a]$. From the second property of being a supremum (see Definition 1.3), there must exist some $n_2 > 1$ such that $x(1) < x(n_2) < [a]$. Now let $\tilde{n}_2 \in \{1, ..., n_2\}$ be such that $x(\tilde{n}_2)$ lies closest to $[a]$, i.e. $x(n) \leq x(\tilde{n}_2) < [a]$ for all $1 \leq n \leq n_2$. Thus we can find $n_3 > n_2$ such that $x(\tilde{n}_2) < x(n_3) < [a]$. By induction we get $n_1 < n_2 < n_3 < ...$ such that:

$$x(n_1) < x(n_2) \leq x(\tilde{n}_2) < x(n_3) \leq x(\tilde{n}_3) < x(n_4) \leq x(\tilde{n}_4) < ...[a],$$

and this is our increasing subsequence.

Now we assume that a belongs to S. Thus we can find $m_1 \geq 1$ such that $x(n) \leq x(m_1) = a$ for all n. We discard the first m_1 elements of our sequence and consider $\{x(n)\}_{n \geq m_1+1} \subset \mathbb{T}$. The new sequence also has a supremum $[b] \leq [a] = x(m_1)$. If b does not belong to the set $\{x(n) : \ n \geq m_1 + 1\}$ then we can reason as before and construct an increasing subsequence with $n_1 = m_1 + 1$. If b belongs to this set, then it must equal some $x(m_2)$ with $m_2 > m_1$; we also have $x(m_2) \leq x(m_1)$. After k steps, then either there exists some set $\{x(n) : \ n \geq m_k + 1\}$ which does not contain its supremum (and we are done), or we can go on to step $k+1$. If this can be continued indefinitely for all k, then we have just constructed a decreasing subsequence $\{x(m_k)\}_{k \geq 1}$ with $m_{k+1} > m_k$ and we are done. □

Proposition 1.20. *Assume that $\{x(n)\}_{n \geq 1} \subset \mathbb{T}$ has the Cauchy property. Then it is strongly convergent.*

Proof. From Lemma 1.18 we know that the sequence is bounded, while from Lemma 1.19 we that that it admits a monotone subsequence $\{x(n_j)\}_{j \geq 1}$. From Lemma 1.15 we know that $\{x(n_j)\}_{j \geq 1}$ has a formal limit $x \in \mathbb{D}$, which due to Lemma 1.13 is also a strong limit. From (1.3) it follows that given $k \geq 1$, there exists some $J_k \geq 1$ such that

$$|x(n_j) - T_j(x)| \leq 10^{-k-1}, \quad \forall j \geq J_k.$$

Because the original sequence has the Cauchy property, given $k \geq 1$ we can find M_{k+1} such that

$$|x(n) - x(m)| \leq 10^{-k-1}, \quad \forall n, m \geq M_{k+1}.$$

Now put $j = N_k := \max\{k + 1, J_k, M_{k+1}\}$ and take $n \geq N_k$. We also have that $n_j \geq j \geq M_{k+1}$ hence:

$$|x(n) - T_n(x)| \leq |x(n) - x(n_j)| + |x(n_j) - T_j(x)| + |T_j(x) - T_n(x)| \leq 3 \cdot 10^{-k-1} < 10^{-k}$$

and we are done. □

The following result will allow us to define addition and multiplication of arbitrary real numbers:

Proposition 1.21. *Assume that $\{x(n)\}_{n \geq 1} \subset \mathbb{T}$ and $\{y(n)\}_{n \geq 1} \subset \mathbb{T}$ have the Cauchy property. Then $u(n) := x(n) + y(n)$ and $w(n) := x(n) \cdot y(n)$ also have the Cauchy property.*

Proof. We only prove that $\{w(n)\}_{n \geq 1}$ is Cauchy. According to Lemma 1.18, both $\{x(n)\}_{n \geq 1}$ and $\{y(n)\}_{n \geq 1}$ are bounded, i.e. there exists $J \in \mathbb{N}$ such that

$$\max\{|x(n)|, |y(n)|\} \leq 10^J, \quad \forall n \in \mathbb{N}.$$

We have the identity:

$$w(n) - w(m) = x(n) \cdot (y(n) - y(m)) + (x(n) - x(m)) \cdot y(m)$$

in which we apply the triangle inequality and get:

$$|w(n) - w(m)| \leq 10^J \cdot (|y(n) - y(m)| + |x(n) - x(m)|), \quad \forall n, m \in \mathbb{N}.$$

There exists M_j such that $|x(n) - x(m)| \leq 10^{-J-k-1}$ if $n, m \geq M_j$, and there exists M_j' such that $|y(n) - y(m)| \leq 10^{-J-k-1}$ if $n, m \geq M_j'$. Hence:

$$|w(n) - w(m)| \leq 10^{-k}, \quad \forall n, m \geq \max\{M_j, M_j'\},$$

and we are done.

□

1.3 Addition and multiplication of real numbers

We can finally define the addition and multiplication of two arbitrary real numbers. Let $[x], [y] \in \mathbb{R}$. We define the sequences $x(n) := T_n(x)$ and $y(n) := T_n(y)$. Then $x(n)$ converges strongly to $[x]$ and $y(n)$ converges strongly to $[y]$. From Lemma 1.17 we know that both $x(n)$ and $y(n)$ are Cauchy while Proposition 1.21 implies that $x(n) \cdot y(n)$ and $x(n) + y(n)$ are also Cauchy, hence according to Proposition 1.20 they have a strong limit. We can therefore make the following definition:

Definition 1.22.

$$[x] + [y] := \text{slim}_{n\to\infty}(T_n(x) + T_n(y)), \quad [x] \cdot [y] := \text{slim}_{n\to\infty}(T_n(x) \cdot T_n(y)).$$

The limits are well defined and do not depend on the choice of the decimal if x and/or y belong to a jump. Also, all the relevant properties of addition and multiplication of terminating decimals can be transferred to arbitrary real numbers. For example, let us prove the commutativity:

Proposition 1.23. *For every* $[x], [y] \in \mathbb{R}$ *we have:*

$$[x] + [y] = [y] + [x], \quad [x] \cdot [y] = [y] \cdot [x].$$

Proof. We have $T_n(x) + T_n(y) = T_n(y) + T_n(x)$ and $T_n(x) \cdot T_n(y) = T_n(y) \cdot T_n(x)$ for all n. We only need to take the strong limit on both sides. □

We also prove the distributivity property:

Proposition 1.24. *For every* $[x], [y], [z] \in \mathbb{R}$ *we have:*

$$([x] + [y]) \cdot [z] = [x] \cdot [z] + [y] \cdot [z].$$

Proof. We know that $[x] + [y] := \text{slim}_{n\to\infty}(T_n(x) + T_n(y))$. Let $w \in D$ be a decimal such that $[w] = [x] + [y]$. Then we also have that $\text{slim}_{n\to\infty}T_n(w) = [x] + [y]$, hence

$$\text{slim}_{n\to\infty}[T_n(w) - (T_n(x) + T_n(y))] = 0.$$

According to the definition:

$$([x] + [y]) \cdot [z] = \text{slim}_{n\to\infty}(T_n(w) \cdot T_n(z)).$$

We have:

$$T_n(w) \cdot T_n(z) = [T_n(w) - (T_n(x) + T_n(y))] \cdot T_n(z) + [T_n(x) + T_n(y)] \cdot T_n(z).$$

Because $T_n(w) - (T_n(x) + T_n(y))$ converges strongly to zero and $T_n(z)$ is bounded, Lemma 1.10 implies that their product converges strongly to zero. Then Lemma 1.9 implies that:

$$\mathrm{slim}_{n\to\infty}\{T_n(w)\cdot T_n(z)\} = \mathrm{slim}_{n\to\infty}\{[T_n(x)+T_n(y)]\cdot T_n(z)\}.$$

But on the right hand side we can apply the distributivity for terminating decimals which leads to:

$$\mathrm{slim}_{n\to\infty}\{T_n(w)\cdot T_n(z)\} = \mathrm{slim}_{n\to\infty}\{T_n(x)\cdot T_n(z) + T_n(y)\cdot T_n(z)\}.$$

Now $T_n(x)\cdot T_n(z)$ converges strongly to $[x]\cdot[z]$ while $T_n(y)\cdot T_n(z)$ converges strongly to $[y]\cdot[z]$. Hence we may replace $T_n(x)\cdot T_n(z)$ with $T_n([x]\cdot[z])$ and $T_n(y)\cdot T_n(z)$ with $T_n([y]\cdot[z])$ and get:

$$\mathrm{slim}_{n\to\infty}\{T_n(w)\cdot T_n(z)\} = \mathrm{slim}_{n\to\infty}\{T_n([x]\cdot[z]) + T_n([y]\cdot[z])\} = [x]\cdot[z]+[y]\cdot[z]$$

and we are done. □

Now we prove the compatibility between the order relation and addition:

Proposition 1.25. *For every $[x],[y],[z]\in\mathbb{R}$ with $[x]<[y]$ we have:*

$$[z]+[x]<[z]+[y].$$

Proof. Because $[x]<[y]$, (1.2) implies that there exists some M such that:

$$10^{-M-2}+T_n(x)\le T_n(y),\quad n\ge M.$$

Hence:

$$10^{-M-2}+T_n(x)+T_n(z)\le T_n(y)+T_n(z),\quad n\ge M.$$

Taking the strong limit on both sides and using Lemma 1.11 we get:

$$10^{-M-2}+[x]+[z]\le[y]+[z].$$

An important remark is that for every real number $[u]$ and $J\in\mathbb{N}$ we have that $[u]<[u]+10^{-J}$. This is because $T_n(u)+10^{-J}$ converges formally to $[u]+10^{-J}$ and its components up to index $-J$ do not change anymore if $n>J$. Hence

$$[x]+[z]<10^{-M-2}+[x]+[z]\le[y]+[z]$$

and the proof is over. □

Finally, we prove the compatibility of multiplication with the order relation:

Proposition 1.26. *For every $[x],[y],[z]\in\mathbb{R}$ with $0<[z]$ and $[x]<[y]$ we have:*

$$[z]\cdot[x]<[z]\cdot[y].$$

Proof. Because $[x]<[y]$, (1.2) implies that there exists some M such that:

$$10^{-M-2}\le T_n(y)-T_n(x),\quad n\ge M.$$

Applying the same idea for $0<[z]$ we obtain some M' such that:

$$10^{-M'-2}\le T_n(z),\quad n\ge M'.$$

We have that

$$T_n(z)\cdot T_n(y)-T_n(z)\cdot T_n(x)=T_n(z)\cdot(T_n(x)-T_n(y)),\quad \forall n\ge 1.$$

Hence:

$$10^{-M-M'-4} \leq T_n(z) \cdot (T_n(y) - T_n(x)) = T_n(z) \cdot T_n(y) - T_n(z) \cdot T_n(x), \quad n \geq \max\{M, M'\},$$

or:

$$T_n(z) \cdot T_n(x) + 10^{-M-M'-4} \leq T_n(z) \cdot T_n(y), \quad n \geq \max\{M, M'\}.$$

Taking the strong limit on both sides and using Lemma 1.11 we get:

$$[z] \cdot [x] < [z] \cdot [x] + 10^{-M-M'-4} \leq [z] \cdot [y]$$

which ends the proof.

□

1.4 Division of real numbers

Let $J \geq 1$ be a natural number. We have the identity:

$$(1 - 10^{-J}) \cdot (1 + 10^{-J} + 10^{-2J} + ... + 10^{-nJ}) = 1 - 10^{-(n+1)J}, \quad n \geq 0. \tag{1.6}$$

The sequence $x(n) = 1 + 10^{-J} + 10^{-2J} + ... + 10^{-nJ}$ converges formally (hence strongly) to the real number given by the decimal

$$1,000...01000..01... := 1, (\overline{000..01})_J$$

where we have exactly $J - 1$ zeros between two consecutive 1's. We denoted in parenthesis the *periodic* part of the decimal, containing $J-1$ zeros and a 1. The length of the period is J. Taking the strong limit in (1.6) we have:

$$(1 - 10^{-J}) \cdot 1, (\overline{000..01})_J = 1, (\overline{000..01})_J \cdot (1 - 10^{-J}) = 1. \tag{1.7}$$

In other words, we have just proved that $(1 - 10^{-J})^{-1} = 1, (\overline{000..01})_J$. Moreover, using that $(10^J - 1) = 10^J \cdot (1 - 10^{-J})$ we have:

$$(10^J - 1)^{-1} = (1 - 10^{-J})^{-1} \cdot 10^{-J} = 1, (000..01)_J \cdot 10^{-J} = 0, (\overline{000..01})_J. \tag{1.8}$$

In what follows we will show how to construct the inverse of any positive natural numbers $q \in \mathbb{N}$, $q \geq 1$. If q is a power of 10, we already know that its inverse is implemented by a translation of the comma.

Now assume that q is not a power of 10. Then we know that it exists exactly one natural number $k \geq 0$ such that $10^k < q < 10^{k+1}$. From the Quotient Remainder Theorem we can write:

$$10^{k+1} = a_1 \cdot q + r_1, \quad 10^{k+2} = a_2 \cdot q + r_2, \quad , ..., \quad 10^{k+j} = a_j \cdot q + r_j, ...$$

where $0 \leq r_j < q$ for all j. Because there are only q different possible values for r_j, then if $j \geq q$ it must happen that at least two remainders are equal to each other. Assume that $r_j = r_i$ with $1 \leq i < j \leq q$. Denote by $d = j - i > 0$. We have:

$$10^{k+i} = a_i q + r_i, \quad 10^{k+j} = a_j q + r_j, \quad 10^{k+i} \cdot (10^d - 1) = (a_j - a_i) \cdot q.$$

The number $a_j - a_i$ can be written as:

$$a_j - a_i = m \cdot (10^d - 1) + r, \quad 0 \leq r < (10^d - 1), \quad m \in \mathbb{N}.$$

Thus:

$$10^{k+i} \cdot (10^d - 1) = (a_j - a_i) \cdot q = [m \cdot (10^d - 1) + r] \cdot q,$$

or:

$$1 = 10^{-k-i} \cdot [m + r \cdot (10^d - 1)^{-1}] \cdot q.$$

Since $(10^d - 1)^{-1} = 0, (\overline{000..1})_d$ and because r has at most d nonzero digits, we have $1 = 10^{-k-i} \cdot [m, (\overline{r})_d] \cdot q$, which implies:

$$q^{-1} = 10^{-k-i} \cdot [m, (\overline{r})_d] \tag{1.9}$$

and this shows that we can compute the multiplicative inverse of any integer.

We are now ready to introduce the rational numbers:

Definition 1.27. *The set of rational numbers is defined as:*

$$\mathbb{Q} := \{[x] \in \mathbb{R} : \ [x] = \pm p \cdot q^{-1}, \quad p, q \in \mathbb{N}, \ q \neq 0\}.$$

Clearly, $\mathbb{Z} \subset \mathbb{Q}$. The following theorem describes the decimal structure of any nonzero rational number.

Theorem 1.28. *A real number $[x] \neq 0$ is rational if and only if $x = \pm 10^K \cdot x_N ... x_0, (\overline{P})$ where $K \in \mathbb{Z}$, $N \geq 0$ and $(\overline{P})$ denotes a periodically repeated sequence of a certain finite length.*

Proof. We only consider positive numbers. Let us first prove that any positive real number which has the form given by the theorem, is rational.

If the periodic sequence $\overline{P}$ only consists of zeros, then the number is either natural if $K \geq 0$ or we can take $p = x_N ... x_0$ and $q = 10^{-K}$ if $K < 0$. Hence we may assume that $\overline{P}$ is a non-trivial sequence of length $l \geq 1$. Moreover, we can also assume that $\overline{P}$ does not start with a zero. If it does, we can multiply with a power of 10 and shift the comma, like in the following example:

$$10^6 \cdot 4, (\overline{00313}) = 10^4 \cdot 400, (\overline{31300}).$$

Hence let $x = 10^K \cdot x_N ... x_0, (\overline{P})$ with $\overline{P} = P = t_{l-1} ... t_0$ and $t_{l-1} \neq 0$. Using (1.8) with $J = l$ we have:

$$0, (\overline{P}) = P \cdot 0, (\overline{00..01})_l = P \cdot (10^l - 1)^{-1},$$

and

$$x_N ... x_0, (\overline{P}) = x_N ... x_0 + P \cdot (10^l - 1)^{-1} = \{x_N ... x_0 \cdot (10^l - 1) + P\} \cdot (10^l - 1)^{-1},$$

hence $x = p \cdot q^{-1}$ with $p = 10^K \cdot \{x_N ... x_0 \cdot (10^l - 1) + P\}$ and $q = 10^l - 1$.

Now let us prove the reversed implication: any rational number has an eventually periodic decimal. We have already shown in (1.9) that this holds true if $p = 1$; now we slightly modify that argument in order to allow $p > 1$. We also assume that p and q have no common divisors (are relatively prime), and q is not a power of 10 because if $q = 10^J$ then $x = p \cdot q^{-1}$ would be a terminating decimal.

Moreover, let us show that we may take $p < q$. Indeed, if $p > q$ then we can write $p = a \cdot q + p'$ with $p' < q$ and a a natural number, hence $p \cdot q^{-1} = a + p' \cdot q^{-1}$. Thus if we can prove the claim for $p' \cdot q^{-1}$, adding a natural number to it will preserve its structure after the comma.

If $p < q$ there exists a unique $k \in \mathbb{N}$ such that $10^k \cdot p < q < 10^{k+1} \cdot p$. Applying again the Quotient Remainder Theorem we have:

$$10^{k+1} \cdot p = a_1 \cdot q + r_1, \ ..., \ 10^{k+j} \cdot p = a_j \cdot q + r_j, ...,$$

where again $0 \leq r_j < q$ for all j, and $1 \leq a_1 < 10$, $10 \leq a_2 < 10^2$ and so on. As before, because q is finite, we must have two equal remainders among the first q; assume without loss of generality that $r_i = r_j$ with $1 \leq i < j \leq q$. Denote by $d = j - i > 0$. Then $10^{k+i} \cdot (10^d - 1) \cdot p = (a_j - a_i) \cdot q$. We write as before

$$a_j - a_i = m \cdot (10^d - 1) + r, \quad 0 \leq r < (10^d - 1), \quad m \in \mathbb{N},$$

which implies:

$$p \cdot q^{-1} = [m \cdot (10^d - 1) + r] \cdot 10^{-k-i} \cdot (10^d - 1)^{-1} = 10^{-k-i} \cdot [m, (\overline{r})_d]$$

and the proof is over.

□

Finally, we can prove that every non-zero real number has a multiplicative inverse. Let $[x] > 0$. The sequence $\{T_n(x)\}_{n\geq 1}$ converges formally to x hence strongly to $[x]$. There exists some $J > 0$ such that $10^{-J} < [x]$ hence we can find some N large enough such that

$$10^{-J} \leq T_n(x), \quad n \geq N.$$

We have $\mathbb{T} \subset \mathbb{Q}$ due to the previous theorem, hence $T_n(x)$ is rational and $(T_n(x))^{-1}$ exists. Due to the above estimate we must have $0 < (T_n(x))^{-1} \leq 10^J$ for all $n \geq N$. Define $y(n) := (T_n(x))^{-1}$ for all $n \geq N$. Let us prove that the sequence $\{y(n)\}_{n\geq N}$ has the Cauchy property. Indeed, we may write:

$$y(m) - y(n) = (T_m(x))^{-1} \cdot (T_n(x) - T_m(x)) \cdot (T_n(x))^{-1}, \quad |y(m) - y(n)| \leq 10^{2J} \cdot |T_n(x) - T_m(x)|.$$

Since $\{T_n(x)\}_{n\geq 1}$ is Cauchy, it implies the same property for $\{y(n)\}_{n\geq N}$. Thus it converges strongly to some real number $[y]$. Therefore, taking the strong limit in the identity $1 = T_n(x) \cdot (T_n(x))^{-1}$ we get:

$$[x] \cdot [y] = 1.$$

If $[x] < 0$ then $[x]^{-1} = -(-[x])^{-1}$.

2 The natural topology of a metric space

Let (X, d) be a metric space. We define the open ball of radius $r > 0$ and center at $a \in X$ the set $B_r(a) := \{x \in X : \ d(x, a) < r\}$.

Given a set $A \subset X$ and $a \in A$, we say that a is an interior point of A if there exists $r > 0$ such that $B_r(a) \subset A$. The set of all interior points of A is denoted by $\mathrm{Int}(A)$. We say that A is an open set if all its points are interior points, i.e. $\mathrm{Int}(A) = A$. By convention, the empty set $\emptyset$ is open.

Lemma 2.1. *Any ball $B_r(a)$ is an open set.*

Proof. Let $x_0 \in B_r(a)$. We have that $d(x_0, a) < r$. Define $r_0 := (r - d(x_0, a))/2 > 0$. Then for all $x \in B_{r_0}(x_0)$ we have that $d(x, x_0) < r_0$ and:

$$d(x, a) \leq d(x, x_0) + d(x_0, a) < (r - d(x_0, a))/2 + d(x_0, a) = (r + d(x_0, a))/2 < r,$$

which shows that $B_{r_0}(x_0) \subset B_r(a)$. Thus $B_r(a)$ has only interior points. □

Lemma 2.2.

(i). Let $\{V_\alpha\}_{\alpha\in\mathcal{F}}$ be an arbitrary collection of open sets. Then $A := \cup_\alpha V_\alpha$ is open.

(ii). Let $\{V_j\}_{j=1}^n$ be a finite collection of open sets. Then $B := \cap_{j=1}^n V_j$ is open.

Proof. We start with (i). Let $a \in \cup_\alpha V_\alpha$. There must exist $\alpha_a \in \mathcal{F}$ such that $a \in V_{\alpha_a}$. Since V_{α_a} is open, there exists $r_a > 0$ such that

$$B_{r_a}(a) \subset V_{\alpha_a} \subset \cup_\alpha V_\alpha = A$$

hence a is an interior point of A.

We continue with (ii). Let $a \in \cap_{j=1}^n V_j$. Thus $a \in V_j$ for all j. Hence there exists $r_j > 0$ such that $B_{r_j}(a) \subset V_j$. Let $r := \min\{r_1, \ldots r_n\} > 0$. Thus $B_r(a) \subset B_{r_j}(a) \subset V_j$ for all j, hence $B_r(a) \subset B$ and we are done.

□

We say that a set $A \subset X$ is closed if $A^c := \{x \in X : x \notin A\}$ is open. Given a set $B \subset X$ and $b \in X$, we say that b is an adherent point of B if there exists a sequence $\{x_n\}_{n\geq 1} \subset B$ such that $x_n \in B_{\frac{1}{n}}(b)$ (hence $\lim_{n\to\infty} x_n = b$). The set of all adherent points of B is denoted by $\overline{B}$.

Theorem 2.3. *Let $B \subset X$. Then $B \subset \overline{B}$. Moreover, $B = \overline{B}$ if and only if B is closed.*

Proof. If $a \in B$ we can define the constant sequence $x_n = a \in B$ which converges to a, thus $a \in \overline{B}$ and $B \subset \overline{B}$.

Now assume that $B = \overline{B}$. We want to prove that B is closed, i.e. B^c is open. Let $a \in B^c = \overline{B}^c$. Then a is not an adherent point, which means that there exists $\epsilon > 0$ such that no point of B lies in the ball $B_\epsilon(a)$. In other words, $B_\epsilon(a) \subset B^c$, hence B^c is open.

Now assume that B is closed. We want to prove that $B = \overline{B}$. Assume that this is not true; it would imply the existence of a point $b \in \overline{B}$ such that $b \in B^c$. Since B^c is open, there exists $\epsilon > 0$ such that $B_\epsilon(b) \subset B^c$, i.e. $B_\epsilon(b) \cap B = \emptyset$. But this is incompatible with $b \in \overline{B}$.

□

3 Compact and sequentially compact sets

Definition 3.1. *Let A be a subset of a metric space (X, d). Let $\mathcal{F}$ be an arbitrary set of indices, and consider the family of sets $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$, where each $\mathcal{O}_\alpha \subseteq X$ is open. This family is called an open covering of A if $A \subseteq \bigcup_{\alpha\in\mathcal{F}} \mathcal{O}_\alpha$.*

Definition 3.2. *Assume that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$ is an open covering of A. If $\mathcal{F}'$ is a subset of $\mathcal{F}$, we say that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}'}$ is a subcovering if we still have the property $A \subseteq \bigcup_{\alpha\in\mathcal{F}'} \mathcal{O}_\alpha$. A subcovering is called finite, if $\mathcal{F}'$ contains finitely many elements.*

Definition 3.3. *Let A be a subset of a metric space (X, d). Then we say that A is covered by a finite ϵ-net if there exists a natural number $N_\epsilon < \infty$ and the points $\{\mathbf{x}_1, ..., \mathbf{x}_{N_\epsilon}\} \subseteq A$ such that $A \subseteq \bigcup_{j=1}^{N_\epsilon} B_\epsilon(\mathbf{x}_j)$.*

Definition 3.4. *A subset $A \subset X$ is called compact, if from* any *open covering of A one can extract a* finite *subcovering.*

Definition 3.5. *$A \subset X$ is called sequentially compact if from any sequence $\{x_n\}_{n\geq 1} \subseteq A$ one can extract a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some point $x_\infty \in A$.*

We will see that in metric spaces the two notions of compactness are equivalent.

3.1 Compact implies sequentially compact

We begin with two lemmas:

Lemma 3.6. *Assume that the sequence $\{x_n\}_{n\geq 1} \subset A$ has a range consisting of finitely many points. Then it admits a convergent subsequence whose limit is one of the elements in the range.*

Proof. Assume that the range of the sequence consists of the distinct points $a_1, a_2, \ldots, a_N$. At least one of these points, say a_1, is taken infinitely many times by the sequence elements. Denote by n_k (with $k \geq 1$) the increasing sequence of indices for which $x_{n_k} = a_1$. This defines our convergent subsequence. □

We say that $a \in X$ is an *accumulation point* for a sequence $\{x_n\}_{n\geq 1}$ if for every $\epsilon > 0$ there exists some $x_n \neq a$ such that $x_n \in B_\epsilon(a)$.

Lemma 3.7. *Assume that the sequence $\{x_n\}_{n\geq 1}$ has an accumulation point a. Then $\{x_n\}_{n\geq 1}$ admits a convergent subsequence whose limit is a.*

Proof. Since a is an accumulation point, there exists an index $j \geq 1$ such that $x_j \neq a$ and $x_j \in B_1(a)$. Denote by n_1 the smallest index for which these two properties hold true. Let $r_1 := d(x_{n_1}, a) > 0$. Define n_2 to be the smallest index j for which $x_j \neq a$ and $x_j \in B_{\min\{r_1, \frac{1}{2}\}}(a)$. We must have $n_2 \geq n_1$ since $x_{n_2} \in B_1(a)$; moreover, because $r_2 := d(x_{n_2}, a) < r_1$, we cannot have $n_1 = n_2$. In general, if $k \geq 2$ we define n_k to be the smallest index j for which $x_j \neq a$ and $x_j \in B_{\min\{r_{k-1}, \frac{1}{k}\}}(a)$; moreover, since $r_k := d(x_{n_k}, a) < r_{k-1} < \cdots < r_1$, we must have $n_k > \cdots > n_1$. Then $\{n_k\}_{k\geq 1}$ is a strictly increasing sequence and $0 < d(x_{n_k}, a) < 1/k$. This shows that $\{x_{n_k}\}_{k\geq 1}$ is a subsequence which converges to a. □

Theorem 3.8. *Let $A \subseteq X$ be compact. Then A is sequentially compact.*

Proof. We will assume the opposite, i.e. there exists a sequence $\{x_n\}_{n\geq 1}$ with no convergent subsequence in A. Such a sequence must have an infinite number of distinct points in the range, due to Lemma 3.6. Moreover, we can assume that $\{x_n\}_{n\geq 1}$ has no accumulation points in A (otherwise such a point would be the limit of a subsequence according to Lemma 3.7).

Since no $x \in A$ can be an accumulation point for $\{x_n\}_{n\geq 1}$, there exists $\epsilon_x > 0$ such that the ball $B_{\epsilon_x}(x)$ contains at most one element of the range of $\{x_n\}_{n\geq 1}$.

Clearly, $\{B_{\epsilon_x}(x)\}_{x\in A}$ is an open covering for A. Because A is compact, we can extract a finite subcovering from it:

$$A \subseteq \bigcup_{j=1}^{N} B_{\epsilon_{y_j}}(y_j), \quad N < \infty, \quad \{y_1, \ldots, y_N\} \subset A.$$

Now remember that $\{x_n\}_{n\geq 1} \subseteq A \subseteq \bigcup_{j=1}^{N} B_{\epsilon_{y_j}}(y_j)$ and at the same time, there are at most N distinct points of the range of $\{x_n\}_{n\geq 1}$ in the union $\bigcup_{j=1}^{N} B_{\epsilon_{y_j}}(y_j)$. We conclude that $\{x_n\}_{n\geq 1}$ can only have a finite number of distinct points in its range, thus it must admit a convergent subsequence according to Lemma 3.6. This contradicts our hypothesis. □

3.2 Sequentially compact implies compact

The proof of this fact is slightly more complicated. We need two preparatory results:

Proposition 3.9. *Let A be a sequentially compact set. Then for every $\epsilon > 0$, A can be covered by a finite ϵ-net (see Definition 3.3).*

Proof. If A contains finitely many points, then the proof is obvious, thus we may assume that $\#(A) = \infty$.

Now suppose that there exists some $\epsilon_0 > 0$ such that A cannot be covered by a finite ϵ_0-net. This means that for any N points of A, $\{x_1, ..., x_N\}$, we have:

$$A \not\subset \bigcup_{j=1}^{N} B_{\epsilon_0}(x_j). \tag{3.1}$$

We will now construct a sequence with elements in A which cannot have a convergent subsequence. Choose an arbitrary point $x_1 \in A$. We know from (3.1), for $N = 1$, that we can find $x_2 \in A$ such that $x_2 \in A \setminus B_{\epsilon_0}(x_1)$. This means that $d(x_1, x_2) \geq \epsilon_0$. We use (3.1) again, for $N = 2$, in order to get a point $x_3 \in A \setminus [B_{\epsilon_0}(x_1) \cup B_{\epsilon_0}(x_2)]$. This means that $d(x_3, x_1) \geq \epsilon_0$ and $d(x_3, x_2) \geq \epsilon_0$. Thus we can continue with this procedure and construct a sequence $\{x_n\}_{n\geq 1} \subseteq A$ which obeys

$$d(x_j, x_k) \geq \epsilon_0, \quad j \neq k.$$

In other words, we constructed a sequence in A which cannot have a Cauchy subsequence. This contradicts Definition 3.5. □

The second result states that a compact set is bounded:

Lemma 3.10. *Let A be a (sequentially) compact set. Then there exists a ball which contains A.*

Proof. We know that A can be covered by any finite ϵ-net; choose $\epsilon = 1$. Then here exist N points of A denoted by $\{x_1, ..., x_N\}$ such that $A \subset \bigcup_{j=1}^N B_1(x_j)$.

Denote by $R = \max\{1 + d(x_j, x_k) : \ 1 \leq j, k \leq N\}$. Then we have $B_1(x_j) \subset B_R(x_1)$ for every j, thus $A \subset B_R(x_1)$ and we are done. □

Let us now prove the theorem:

Theorem 3.11. *Assume that $A \subseteq X$ is sequentially compact. Then A is compact.*

Proof. Consider an arbitrary open covering of A:

$$A \subseteq \bigcup_{\alpha \in \mathcal{F}} \mathcal{O}_\alpha.$$

We will show that we can extract a finite subcovering from it.

For every $x \in A$, there exists at least one open set $\mathcal{O}_{\alpha(x)}$ such that $x \in \mathcal{O}_{\alpha(x)}$. Because $\mathcal{O}_{\alpha(x)}$ is open, we can find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{O}_{\alpha(x)}$.

For a fixed x, we define the set

$$E_x := \{r > 0 : \text{ there exists } \alpha \in \mathcal{F} \text{ such that } B_r(x) \subseteq \mathcal{O}_\alpha\} \subset \mathbb{R}.$$

From the above argument we conclude that no E_x is empty. Moreover, if $r \in E_x$, then the open interval $(0, r)$ is included in E_x.

If for some x in A we have an unbounded E_x, it follows that for every $r > 0$ we can find some open set $\mathcal{O}_\alpha$ such that $B_r(x) \subseteq \mathcal{O}_\alpha$. But if r is chosen to be large enough, it will contain the ball we constructed in Lemma 3.10, thus $\mathcal{O}_\alpha$ will also contain A. In this case we found our subcovering, which consists of just one open set.

It follows that we may assume that all the sets E_x are bounded intervals admitting a positive and finite supremum $\sup E_x$. Define $0 < \epsilon_x := \frac{1}{2} \sup E_x < \sup E_x$. Note the important thing that $\epsilon_x \in E_x$. Let us also observe that:

$$A \subseteq \bigcup_{x \in A} B_{\epsilon_x}(x) \subseteq \bigcup_{\alpha \in \mathcal{F}} \mathcal{O}_\alpha. \tag{3.2}$$

The first inclusion is obvious, while the second one follows from the above discussion.

We now need to prove a lemma:

Lemma 3.12. *If A is sequentially compact, then*

$$\inf_{x \in A} \epsilon_x =: 2\epsilon_0 > 0.$$

In other words, there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq B_{\epsilon_x}(x)$, for every $x \in A$.

Proof. Assume that $\inf_{x \in A} \epsilon_x = 0$. This implies that there exists a sequence $\{x_n\}_{n \geq 1} \subseteq A$ such that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. Since A is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k \geq 1}$ which converges to a point $x_0 \in A$, i.e.

$$\lim_{k \to \infty} x_{n_k} = x_0. \tag{3.3}$$

Because x_0 belongs to A, we can find an open set $\mathcal{O}_{\alpha(x_0)}$ which contains x_0, thus we can find $\epsilon_1 > 0$ such that

$$B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}. \tag{3.4}$$

Now (3.3) implies that there exists $K > 0$ large enough such that:

$$d(x_{n_k}, x_0) \leq \epsilon_1/4, \quad \text{whenever} \quad k > K. \tag{3.5}$$

If y belongs to $B_{\epsilon_1/4}(x_{n_k})$ (i.e. $d(y, x_{n_k}) < \epsilon_1/4$), then the triangle inequality implies (use also (3.5)):

$$d(y, x_0) \leq d(y, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon_1/2 < \epsilon_1, \quad k > K.$$

But this shows that we must have $y \in B_{\epsilon_1}(x_0)$, or:

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K. \tag{3.6}$$

Thus we got the inclusion

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K,$$

which shows that $\epsilon_1/4$ must be less or equal than $2\epsilon_{x_{n_k}}$, or $\epsilon_1/8 \leq \epsilon_{x_{n_k}}$, for every $k > K$. But this is in contradiction with the fact that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. □

Finishing the proof of Theorem 3.11. We now use Proposition 3.9, and find a finite ϵ_0-net for A. Thus we can choose $\{y_1, ...y_N\} \subseteq A$ such that

$$A \subseteq \bigcup_{n=1}^{N} B_{\epsilon_0}(y_n) \subseteq \bigcup_{n=1}^{N} B_{\epsilon_{y_n}}(y_n) \subseteq \bigcup_{n=1}^{N} \mathcal{O}_n,$$

where $\mathcal{O}_n$ is one of the possibly many open sets which contain $B_{\epsilon_{y_n}}(y_n)$. We have thus extracted our finite subcovering of A and the proof of the theorem is over. □

3.3 The Bolzano-Weierstrass Theorem

We start with the case in which the metric space is $\mathbb{R}$ with the Euclidean distance.

Theorem 3.13. *Let $\{x_n\} \subset \mathbb{R}$ be a bounded real sequence, i.e. there exists $M \geq 0$ such that $|x_n| \leq M$ for all $n \geq 1$. Then there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ and some $s \in \mathbb{R}$ such that $\lim_{k\to\infty} x_{n_k} = s$.*

Proof. We have that $-M \leq x_n \leq M$ for all n. Define by $a_1 := -M$ and $b_1 := M$. Since either $-M \leq x_n \leq 0$ or $0 \leq x_n \leq M$ for any given n, it follows that at least one of the two intervals $[-M, 0]$ and $[0, M]$ must contain x_n for infinitely many different values of n. If there are infinitely many indices such that $x_n \in [-M, 0]$, then define $a_2 := a_1$ and $b_2 := (a_1 + b_1)/2$. If this is not true, then define $a_2 := (a_1 + b_1)/2$ and $b_2 := b_1$. If the first case holds true, we define n_1 to be the smallest index n for which $-M = a_2 \leq x_n \leq b_2 = 0$, while if the second case is true, we define n_1 to be the smallest index n for which $0 = a_2 \leq x_n \leq b_2 = M$.

In either case, we know that there exist infinitely many indices n such that $a_2 \leq x_n \leq b_2$, and n_1 is the smallest of them. If the interval $[a_2, (a_2 + b_2)/2]$ contains x_n for infinitely many values of n, then define $a_3 := a_2$ and $b_3 := (a_2 + b_2)/2$. If this is not true, then define $a_3 := (a_2 + b_2)/2$ and $b_3 := b_2$; the interval $[a_3, b_3]$ will thus contain x_n infinitely many times. We can thus choose n_2 to be the smallest index $n > n_1$ for which $a_3 \leq x_n \leq b_3$. By induction, for a given $k \geq 1$, we can construct $n_k > n_{k-1} > \cdots > n_1$ such that $a_{k+1} \leq x_{n_k} \leq b_{k+1}$, where either $a_{k+1} := a_k$ and $b_{k+1} := (a_k + b_k)/2$ (if the interval $[a_k, (a_k + b_k)/2]$ contains x_n infinitely many times), or $a_{k+1} := (a_k + b_k)/2$ and $b_{k+1} := b_k$ otherwise. By construction we have that $a_k \leq a_{k+1}$ and $b_{k+1} \leq b_k$ for all k. Moreover, $a_k \leq b_k$ for all k, and in particular $a_k \leq b_1 = M$ and $a_1 = -M \leq b_k$. By induction, we can also prove that $b_k - a_k = (b_1 - a_1)/2^{k-1}$.

Thus $\{a_k\}_{k\geq 1}$ is increasing and bounded from above, hence it converges to $\alpha := \sup_{k\geq 1} a_k$. The sequence $\{b_k\}_{k\geq 1}$ is decreasing and bounded from below, thus it converges to $\beta := \inf_{k\geq 1} b_k$. By taking the limit $k \to \infty$ in the equality $b_k - a_k = (b_1 - a_1)/2^{k-1}$ we conclude that $\alpha = \beta$. Since $a_k \leq x_{n_k} \leq b_k$, by the comparison theorem it follows that $\{x_{n_k}\}_{k\geq 1}$ is convergent and has the limit $s := \alpha = \beta$.

□

We can generalize this result to $\mathbb{R}^d$, with $d \geq 2$. Without loss of generality, assume that $d = 2$; the general case follows by induction. If $\mathbf{x} = [u, v] \in \mathbb{R}^2$, then we define $||\mathbf{x}|| = \sqrt{u^2 + v^2}$. Clearly,

$\max\{|u|, |v|\} \leq ||\mathbf{x}|| \leq |u| + |v|$. The Euclidean distance between two vectors $\mathbf{x} = [u_1, v_1]$ and $\mathbf{y} = [u_2, v_2]$ is given by $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$. It is easy to check that $d(\mathbf{x}, \mathbf{y}) \leq |u_1 - u_2| + |v_1 - v_2|$.

Now assume that the sequence $\{\mathbf{x}_n\}_{n\geq 1} \subset \mathbb{R}^2$ is bounded, i.e. there exists $M \geq 0$ such that $||\mathbf{x}_n|| \leq M$ for all n. We denote the components of $\mathbf{x}_n$ with $[u_n, v_n]$. The real sequence $\{u_n\}_{n\geq 1} \subset \mathbb{R}$ is also bounded by M, thus from Theorem 3.13 it follows that we can find a subsequence $\{u_{n_k}\}_{k\geq 1}$ which is convergent to some $t \in \mathbb{R}$, i.e. $\lim_{k\to\infty} u_{n_k} = t$. Define $z_k := v_{n_k}$; then $\{z_k\}_{k\geq 1}$ is also bounded by M and according to Theorem 3.13 we can find a subsequence $\{z_{k_j}\}_{j\geq 1}$ which is convergent to some $s \in \mathbb{R}$, i.e. $\lim_{j\to\infty} z_{k_j} = s$. Thus we have that $v_{n_{k_j}}$ converges to s while $u_{n_{k_j}}$ still converges to t, as a subsequence of the convergent sequence $\{u_{n_k}\}_{k\geq 1}$.

Define $\mathbf{y} := [t, s]$. We have $0 \leq d(\mathbf{x}_{n_{k_j}}, \mathbf{y}) \leq |u_{n_{k_j}} - t| + |v_{n_{k_j}} - s|$ for all $j \geq 1$, which shows that $\mathbf{y}$ is the limit of $\{\mathbf{x}_{n_{k_j}}\}_{j\geq 1}$.

3.4 The Heine-Borel Theorem

Lemma 3.14. *Let A be a compact set in a metric space (X, d). Then A is bounded and closed.*

Proof. We already know that a compact set A is bounded (see Lemma 3.10). Let us prove that it is closed. Assume it is not. According to Theorem 2.3 it means that there exists an adherent point $a \in \overline{A}$ which does not belong to A. Being an adherent point, there exists a sequence $\{x_n\}_{n\geq 1} \subset A$ which converges to a, thus all of its subsequences must converge to the same limit. Since A is (sequentially) compact, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some point of A, which has to be a. This contradicts the fact that $a \notin A$.

□

Theorem 3.15. *Consider $\mathbb{R}^d$ with the Euclidean distance. In this metric space, a set A is (sequentially) compact if and only if A is both bounded and closed.*

Proof. The previous lemma showed that a compact set is always bounded and closed; this fact holds for all metric spaces, not just for the Euclidean ones.

If the space is Euclidean, then we can also show the reversed implication. Assume that A is bounded and consider an arbitrary sequence $\{x_n\}_{n\geq 1} \subset A$. The Bolzano-Weierstrass theorem implies the existence of a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some point $a \in \mathbb{R}^d$. Thus $a \in \overline{A}$, and due to Theorem 2.3 we know that $A = \overline{A}$, thus $a \in A$. This proves that A is sequentially compact, therefore compact.

□

4 Continuous functions on metric spaces

Let (X, d) and (Y, ρ) be two metric spaces. If $A \subset X$, the image of A through f is the set

$$f(A) := \{y \in Y : \text{ there exists } x_y \in A \text{ such that } f(x_y) = y\} \subset Y.$$

If $B \subset Y$ the preimage of B through f is the set

$$f^{-1}(B) := \{x \in X : \text{ such that } f(x) \in B\} \subset X.$$

Note that the notation $f^{-1}(B)$ does not imply that f is invertible.

Lemma 4.1. *If $A_1 \subset A_2 \subset X$ and $B_1 \subset B_2 \subset Y$ then $f(A_1) \subset f(A_2)$ and $f^{-1}(B_1) \subset f^{-1}(B_2)$.*

Proof. We only prove the first inclusion. Assume that $y \in f(A_1)$. Then there exists $x_y \in A_1$ such that $f(x_y) = y$. But at the same time $x_y \in A_2$, hence $y \in f(A_2)$. □

A map $f : X \to Y$ is said to be continuous at a point $a \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$B_\delta(a) \subset f^{-1}(B_\epsilon(f(a))), \tag{4.1}$$

which implies that $f(B_\delta(a)) \subset B_\epsilon(f(a))$. The function is continuous on X if it is continuous at all the points of X.

Theorem 4.2. *A function between two metric spaces $f : X \to Y$ is continuous on X if and only if for every nonempty open set $V \subset Y$ we have that $f^{-1}(V)$ is open in X.*

Proof. First we assume that f is continuous on X. Let V a nonempty open set in Y. If $f^{-1}(V)$ is empty then we know that it is open. Otherwise, let $a \in f^{-1}(V)$. Thus $f(a) \in V$. Since V is open, $f(a)$ is an interior point of V, thus there exists $\epsilon > 0$ such that $B_\epsilon(f(a)) \subset V$. Applying Lemma 4.1 we get that $f^{-1}(B_\epsilon(f(a))) \subset f^{-1}(V)$. But from (4.1) it follows that $B_\delta(a) \subset f^{-1}(V)$, thus a is an interior point.

We now assume that f returns any nonempty open set V of Y in an open set $f^{-1}(V)$ of X. Fix $a \in X$. Let $\epsilon > 0$ and consider the ball $B_\epsilon(f(a))$. Lemma 2.1 implies that $V = B_\epsilon(f(a))$ is open in Y. Thus $f^{-1}(B_\epsilon(f(a)))$ must be open in X. Since $a \in f^{-1}(B_\epsilon(f(a)))$, it must be an interior point. Thus there exists $\delta > 0$ such that $B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$, which shows that f is continuous at a.

□

Let (X, d) and (Y, ρ) be two metric spaces and consider a subset $A \subset X$. We can organize A as a metric space with the natural distance d_A induced by d. We say that the map $f : A \mapsto Y$ is continuous on A if it is continuous between the metric spaces (A, d_A) and (Y, ρ).

We say that $f : A \mapsto Y$ is sequentially continuous at a point $a \in A$ if for every sequence $\{x_n\}_{n\geq 1} \subset A$ which converges to a we have that $\{f(x_n)\}_{n\geq 1} \subset Y$ converges to $f(a)$. We say that $f : A \mapsto Y$ is sequentially continuous on A if it is sequentially continuous at all points of A.

Theorem 4.3. *With the above notation, consider a map $f : A \mapsto Y$. Then f is continuous on A if and only if it is sequentially continuous on A.*

Proof. First, assume that f is continuous at $a \in A$. Consider any sequence $\{x_n\}_{n\geq 1} \subset A$ which converges to a. From (4.1) we know that for every $\epsilon > 0$ we have that $\rho(f(x_n), f(a)) < \epsilon$ if $d(x_n, a) < \delta$. But the second inequality holds if n is larger than some $N_\delta \geq 1$. Thus $\{f(x_n)\}_{n\geq 1} \subset Y$ converges to $f(a)$.

Second, assume that f is sequentially continuous at $a \in A$. We will show that f must be continuous at a. Suppose this is not true: it means that there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ we have that $B_\delta(a) \not\subset f^{-1}(B_{\epsilon_0}(f(a)))$. By letting $\delta = 1/n$ for all $n \geq 1$, we can find a point $x_n \in B_{\frac{1}{n}}(a)$ such that $f(x_n) \notin B_{\epsilon_0}(f(a))$, or $\rho(f(x_n), f(a)) \geq \epsilon_0$. In this way we constructed a sequence $\{x_n\}_{n\geq 1} \subset A$ which converges to a while $\{f(x_n)\}_{n\geq 1}$ does not converge to $f(a)$, contradiction.

□

Theorem 4.4. *With the above notation, consider a continuous map $f : A \mapsto Y$ where $A \subset X$ is compact. Then $f(A)$ is compact.*

Proof. We show that $f(A)$ is sequentially compact. Let $\{y_n\}_{n\geq 1} \subset f(A)$ be an arbitrary sequence. There exists $\{x_n\}_{n\geq 1} \subset A$ such that $f(x_n) = y_n$. Since A is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k\geq 1} \subset \{x_n\}_{n\geq 1}$ which converges to some point $a \in A$. But f is sequentially continuous at a, hence $y_{n_k} = f(x_{n_k})$ converges to $f(a) \in f(A)$. Hence $f(A)$ is sequentially compact.

□

The next lemma recalls a general result which says that real continuous functions defined on compact sets attain their extremal values.

Lemma 4.5. *Let (X, d) be a metric space and let $H \subset X$ be a compact set. Let $f : H \mapsto \mathbb{R}$ be continuous on H. Then there exist x_m and x_M in H such that $f(x_M) = \sup_{x\in H} f(x)$ and $f(x_m) = \inf_{x\in H} f(x)$.*

Proof. We only prove this for $\sup_{x\in H} f(x)$. Let $B := f(H) \subset \mathbb{R}$. Let us show that there exists a sequence $\{x_n\}_{n\geq 1} \subset H$ such that $\lim_{n\to\infty} f(x_n) = \sup_{x\in H} f(x) = \sup(B)$.

Since B is compact, it is bounded. Thus $\sup(B) = \sup_{x\in H} f(x) < \infty$. For every $n \geq 1$ we know that $\sup(B) - 1/n$ is not an upper bound for B, thus there must exist $x_n \in H$ such that $\sup(B) - 1/n < f(x_n) \leq \sup(B)$. Thus $\lim_{n\to\infty} f(x_n) = \sup(B)$.

Because H is compact, we can find a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges towards some point $a \in H$. Since f is continuous, we have that $\lim_{k\to\infty} f(x_{n_k}) = f(a)$. Since $\{f(x_{n_k})\}_{k\geq 1}$ is a subsequence of the convergent sequence $\{f(x_n)\}_{n\geq 1}$, we must have $f(a) = \sup(B)$. Thus we can choose x_M to be a.

□

We say that $f : A \mapsto Y$ is uniformly continuous on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ as soon as $x, y \in A$ and $d(x, y) < \delta$. Clearly, if f is uniformly continuous on A then it is also continuous. The next result gives sufficient conditions for the reciprocal statement:

Lemma 4.6. *Let (X, d) and (Y, ρ) be two metric spaces and let $H \subset X$ be a compact set. Let $f : H \mapsto Y$ be continuous on H. Then f is uniformly continuous on H.*

Proof. Assume that the conclusion is false. Then there exists $\epsilon_0 > 0$ such that regardless how large $n \geq 1$ is, we may find two points x_n and y_n in H which obey $d(x_n, y_n) < \frac{1}{n}$ and $\rho(f(x_n), f(y_n)) \geq \epsilon_0$. Since H is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some point $a \in H$. Because $d(y_{n_k}, a) \leq \frac{1}{k} + d(x_{n_k}, a)$ for all $k \geq 1$, it follows that y_{n_k} also converges to a. The function f is sequentially compact at a, thus both $f(x_{n_k})$ and $f(y_{n_k})$ converge to $f(a)$. In particular, this contradicts our assumption that $\rho(f(x_{n_k}), f(y_{n_k})) \geq \epsilon_0$ for all k.

□

5 Banach's fixed point theorem

Definition 5.1. *Let (X, d) be a metric space. A map $F : X \to X$ is called a contraction if there exists $\alpha \in [0, 1)$ such that:*

$$d(F(x), F(y)) \leq \alpha d(x, y), \quad \forall x, y \in X. \tag{5.2}$$

A point $x \in X$ is a fixed point for F if $F(x) = x$.

Theorem 5.2. *Let (X, d) be a complete metric space and $F : X \to X$ a contraction. Then F has a unique fixed point.*

Proof. Vi start by showing uniqueness. Assume that there exist $a, b \in X$ such that $F(a) = a$ and $F(b) = b$. Then (5.2) implies that

$$0 \leq d(a, b) = d(F(a), F(b)) \leq \alpha d(a, b), \quad (1 - \alpha)d(a, b) \leq 0,$$

i.e. $d(a, b) = 0$ and $a = b$.

Now let us construct such a fixed point. Consider the sequence $\{y_n\}_{n\geq 1} \subset X$, where y_1 is arbitrary and $y_n := F(y_{n-1})$ for every $n \geq 2$. If $y_2 = F(y_1) = y_1$ then y_1 is our fixed point and we are done; hence we may assume that $d(y_2, y_1) > 0$.

We will now show two things:

(i). The sequence is Cauchy in X, thus convergent to a limit y_∞ because we assumed X to be complete;

(ii). y_∞ is a fixed point for F.

Let us start with (i). For every $\epsilon > 0$ we will construct $N(\epsilon) > 0$ such that for all $p \geq q \geq N(\epsilon)$ we have $d(y_q, y_p) < \epsilon$. In other words:

$$d(y_q, y_{q+k}) < \epsilon, \quad \forall k \geq 0, \quad \forall q \geq N(\epsilon). \tag{5.3}$$

If $k \geq 1$, the triangle inequality implies:

$$\begin{aligned} d(y_q, y_{q+k}) &\leq d(y_q, y_{q+1}) + d(y_{q+1}, y_{q+k}) \\ &\leq d(y_q, y_{q+1}) + d(y_{q+1}, y_{q+2}) + d(y_{q+2}, y_{q+k}) \\ &\leq \sum_{i=0}^{k-1} d(y_{q+i}, y_{q+i+1}). \end{aligned} \tag{5.4}$$

For every $n \geq 1$ we have:

$$d(y_n, y_{n+1}) = d(F(y_{n-1}), F(y_n)) \leq \alpha d(y_{n-1}, y_n) \leq \cdots \leq \alpha^{n-1} d(y_1, y_2), \quad \forall n \geq 1.$$

Thus $d(y_{q+i}, y_{q+i+1}) \leq \alpha^{q+i-1} d(y_1, y_2)$ for all $q \geq 1$ and $i \geq 0$. Together with (5.4), this implies:

$$d(y_q, y_{q+k}) \leq \alpha^{q-1} d(y_1, y_2)(1 + \cdots + \alpha^{k-1}) \leq \frac{\alpha^{q-1}}{1-\alpha} d(y_1, y_2), \quad \forall k \geq 1.$$

Because $\alpha < 1$, we have $\lim_{q\to\infty} \alpha^q = 0$. In other words, we can find some large enough $N(\epsilon)$ such that for every $q \geq N(\epsilon)$ to have

$$\alpha^q < \epsilon \, \frac{\alpha(1-\alpha)}{d(y_1, y_2)}$$

and (5.3) follows. We conclude that there exists $y \in X$ such that

$$\lim_{n\to\infty} d(y_n, y) = 0. \tag{5.5}$$

Now we prove (ii). For every $n \geq 1$ we have:

$$d(F(y), y) \leq d(F(y), F(y_n)) + d(F(y_n), y).$$

But $d(F(y), F(y_n)) \leq \alpha d(y, y_n) \to 0$ and $d(F(y_n), y) = d(y_{n+1}, y) \to 0$ when $n \to \infty$, thus $d(F(y), y) = 0$ and $F(y) = y$. □

6 Local existence and uniqueness for first order ODE's

We start with some general facts about functional spaces.

6.1 Spaces of bounded/continuous functions

Let Y be a real vector space. The map $||\cdot|| : Y \mapsto \mathbb{R}_+$ is called a norm if it fulfills three conditions:

(1). $||y|| = 0$ iff $y = 0$;

(2). $||\lambda y|| = |\lambda| \, ||y||$, for all $\lambda \in \mathbb{R}$ and $y \in Y$;

(3). $||y + z|| \leq ||y|| + ||z||$ for all $y, z \in Y$.

Proposition 6.1. *Let (A, d) be a metric space, $(Y, ||\cdot||)$ a normed space, and H an arbitrary non-empty subset of A. We define*

$$B(H;Y) := \{f : H \to Y : \sup_{x\in H} ||f(x)|| < \infty\}.$$

Define $||\cdot||_\infty : B(H;Y) \to \mathbb{R}_+$, $||f||_\infty := \sup_{x\in H} ||f(x)||$. Then the space $(B(H;Y), ||\cdot||_\infty)$ is a normed space, and the map $d_\infty(f, g) := ||f - g||_\infty$ defines a metric.

Proof. We first check the three conditions for being a norm. We have $||f||_\infty = 0$ if and only if $||f(x)|| = 0$ for all $x \in H$, which is equivalent with $f = 0$ and this proves (1).

If $\lambda = 0$, (2) follows from (1). Hence we may assume that $\lambda \neq 0$. Since $||\lambda f(x)|| = |\lambda|\, ||f(x)||$ for all x we have

$$||\lambda f(x)|| = |\lambda|\, ||f(x)|| \leq |\lambda| \sup_{y\in H} ||f(y)|| = |\lambda|\, ||f||_\infty$$

which shows that $|\lambda|\, ||f||_\infty$ is an upper bound for all the numbers of the form $||\lambda f(x)||$. Hence:

$$||\lambda f||_\infty = \sup_{x\in H} ||\lambda f(x)|| \leq |\lambda|\, ||f||_\infty.$$

On the other hand,

$$||f(x)|| = \frac{1}{|\lambda|} ||\lambda f(x)|| \leq \frac{1}{|\lambda|} ||\lambda f||_\infty$$

which means that $\frac{1}{|\lambda|}||\lambda f||_\infty$ is an upper bound for all the numbers of the form $||f(x)||$. Hence:

$$||f||_\infty \leq \frac{1}{|\lambda|} ||\lambda f||_\infty, \quad \text{or} \quad |\lambda|\, ||f||_\infty \leq ||\lambda f||_\infty.$$

Thus $|\lambda|\, ||f||_\infty = ||\lambda f||_\infty$ and (2) is proved.

Finally, let us prove the triangle inequality (3). Fix $f, g \in B(H;Y)$ and for every $x \in H$ we apply the triangle inequality in $(Y, ||\cdot||)$:

$$||f(x) + g(x)|| \leq ||f(x)|| + ||g(x)|| \leq ||f||_\infty + ||g||_\infty.$$

Thus $||f||_\infty + ||g||_\infty$ is an upper bound for the set $\{||f(x) + g(x)|| : x \in H\}$, hence

$$\sup_{x\in H} ||f(x) + g(x)|| = ||f + g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

Note that $d_\infty(f, g) := ||f - g||_\infty$ is the metric induced by the norm. □

Proposition 6.2. *Denote by $C(H;Y)$ the subset of $B(H;Y)$ where the functions are also continuous. Assume that $(Y, ||\cdot||)$ is a Banach space (a complete normed space). Then $(C(H;Y), ||\cdot||_\infty)$ is a Banach space, too.*

Proof. We need to prove that every Cauchy sequence is convergent. Assume that $\{f_n\}_{n\geq 1} \subset C(H;Y)$ is Cauchy, i.e. for every $\epsilon > 0$ one can find $N_C(\epsilon) > 0$ such that $||f_p - f_q||_\infty < \epsilon$ if $p, q > N_C(\epsilon)$. We have to show that the sequence has a limit f which belongs to $C(H;Y)$.

We first construct f. For every $x \in H$ we consider the sequence $\{f_n(x)\}_{n\geq 1} \subset Y$. Note the conceptual difference between $\{f_n(x)\}_{n\geq 1}$ (a sequence of vectors from Y) and $\{f_n\}_{n\geq 1}$ (a sequence of functions from $C(H;Y)$). Because

$$||f_p(x) - f_q(x)|| \leq ||f_p - f_q||_\infty$$

we have that the sequence $\{f_n(x)\}_{n\geq 1}$ is Cauchy in Y. Since Y is complete, then $\{f_n(x)\}_{n\geq 1}$ must have a limit in Y. We denote it with $f(x)$. Moreover, since $\{f_n\}_{n\geq 1}$ is Cauchy it must be bounded, i.e. there exists a constant $M < \infty$ such that $||f_n||_\infty \leq M < \infty$ for all $n \geq 1$. The triangle inequality gives:

$$||f(x)|| \leq ||f(x) - f_n(x)|| + ||f_n(x)|| \leq ||f(x) - f_n(x)|| + M,$$

and after taking the limit $n \to \infty$ we get:

$$||f(x)|| \leq M, \quad \forall x \in H,$$

hence $||f||_\infty \leq M < \infty$.

The function f we have just constructed is our candidate for the limit in the norm $||\cdot||_\infty$. Now we want to show that for every $\epsilon > 0$ we can find $N_1(\epsilon) > 0$ so that:

$$\sup_{x \in H} ||f(x) - f_n(x)|| < \epsilon \quad \text{whenever} \quad n > N_1(\epsilon). \tag{6.1}$$

In order to do that, take an arbitrary point $x \in H$. For every $p, n \geq 1$ we have

$$\begin{aligned} ||f(x) - f_n(x)|| &\leq ||f(x) - f_p(x)|| + ||f_p(x) - f_n(x)|| \\ &\leq ||f(x) - f_p(x)|| + ||f_p - f_n||_\infty. \end{aligned} \tag{6.2}$$

If we choose $n, p > N_C(\epsilon/2)$, then we have $||f_p - f_n||_\infty < \epsilon/2$ and

$$||f(x) - f_n(x)|| \leq ||f(x) - f_p(x)|| + \epsilon/2, \quad \forall n, p > N_C(\epsilon/2).$$

But the above left hand side does not depend on p, thus if we take $p \to \infty$ on the right hand side, we get:

$$||f(x) - f_n(x)|| \leq \epsilon/2 < \epsilon, \quad n > N_C(\epsilon/2). \tag{6.3}$$

Note that this inequality holds true *for every* x. This means that $\epsilon/2$ is an upper bound for the set $\{||f(x) - f_n(x)|| : \ x \in H\}$, hence (6.1) holds true with $N_1(\epsilon) = N_C(\epsilon/2)$.

Until now we have proved that f is bounded. Now we want to prove that f is a continuous function on H. Fix some point $a \in H$. Choose $\epsilon > 0$. We define $n_1 := N_1(\epsilon/3) = N_C(\epsilon/6)$. Because f_{n_1} is continuous at a, we can find $\delta(\epsilon, a) > 0$ so that for every $x \in H$ with $d(x, a) < \delta$ we have $||f_{n_1}(x) - f_{n_1}(a)|| < \epsilon/3$. Thus if $x \in H$ with $d(x, a) < \delta$ we have:

$$\begin{aligned} ||f(x) - f(a)|| &\leq ||f(x) - f_{n_1}(x)|| + ||f_{n_1}(x) - f_{n_1}(a)|| + ||f_{n_1}(a) - f(a)|| \\ &< 2||f - f_{n_1}||_\infty + ||f_{n_1}(x) - f_{n_1}(a)|| < \epsilon. \end{aligned} \tag{6.4}$$

Since a is arbitrary, we can conclude that f is continuous on H, thus belongs to $C(H; Y)$. Therefore the space is complete. □

6.2 The main theorem

Let U be an open set in $\mathbb{R}^d$, $d \geq 1$, and $I \subset \mathbb{R}$ an open interval. Assume that there exist $\mathbf{y}_0 \in U$ and $r_0, \delta_0 > 0$ such that $\overline{B_{r_0}(\mathbf{y}_0)} \subset U$ and $[t_0 - \delta_0, t_0 + \delta_0] \subset I$.

We consider a continuous function $\mathbf{f} : I \times U \to \mathbb{R}^d$ for which there exists $L > 0$ such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|, \quad \forall t \in [t_0 - \delta_0, t_0 + \delta_0], \quad \forall \mathbf{x}, \mathbf{y} \in \overline{B_{r_0}(\mathbf{y}_0)}. \tag{6.5}$$

We define the set $H_0 := [t_0 - \delta_0, t_0 + \delta_0] \times \overline{B_{r_0}(\mathbf{y}_0)} \subset \mathbb{R}^{d+1}$. Since H_0 is bounded and closed, it must be compact.

Using the triangle inequality we obtain:

$$|\ ||\mathbf{f}(t, \mathbf{x})|| - ||\mathbf{f}(s, \mathbf{y})||\ | \leq ||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(s, \mathbf{y})||$$

which shows that the continuity of $\mathbf{f}$ implies continuity for $||\mathbf{f}||$. Since $||\mathbf{f}||$ is a real valued continuous function defined on a compact set, according to Lemma 4.5 we can find $M < \infty$ such that

$$\sup_{[t, \mathbf{x}] \in H_0} \|\mathbf{f}(t, \mathbf{x})\| =: M < \infty. \tag{6.6}$$

Theorem 6.3. *Consider the initial value problem:*

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \tag{6.7}$$

Define $\delta_1 := \min\{\delta_0, r_0/M, 1/L\}$. *Then there exists a solution* $\mathbf{y} : (t_0 - \delta_1, t_0 + \delta_1) \mapsto \overline{B_{r_0}(\mathbf{y}_0)}$, *which is unique.*

Proof. Take some $0 < \delta < \delta_1$ and define the compact interval $K := [t_0 - \delta, t_0 + \delta] \subset \mathbb{R}$. Then any continuous function $\phi : K \to \mathbb{R}^d$ is automatically bounded, and since the Euclidean space $Y = \mathbb{R}^d$ is a Banach space, we can conclude from Proposition 6.2 that the space $(C(K; \mathbb{R}^d), d_\infty)$ of continuous functions defined on the compact K with values in $\mathbb{R}^d$ is a complete metric space.

Define

$$X := \{\mathbf{g} \in C(K; \mathbb{R}^d) : \quad \mathbf{g}(t) \in \overline{B_{r_0}(\mathbf{y}_0)}, \ \forall t \in K\}. \tag{6.8}$$

Lemma 6.4. *The metric space* (X, d_∞) *is complete.*

Proof. Consider a Cauchy sequence $\{\mathbf{g}_n\}_{n \geq 1} \subset X$. Because $(C(K; \mathbb{R}^d), d_\infty)$ is complete, we can find $\mathbf{g} \in C(K; \mathbb{R}^d)$ such that $\lim_{n\to\infty} d_\infty(\mathbf{g}_n, \mathbf{g}) = 0$. Thus for every $t \in K$ we have

$$\mathbf{g}(t) = \lim_{n\to\infty} \mathbf{g}_n(t), \quad \lim_{n\to\infty} \|\mathbf{g}_n(t) - \mathbf{g}(t)\| = 0.$$

Since by assumption $\|\mathbf{g}_n(t) - \mathbf{y}_0\| \leq r_0$ for all t and n, we have

$$\|\mathbf{g}(t) - \mathbf{y}_0\| = \lim_{n\to\infty} \|\mathbf{g}_n(t) - \mathbf{y}_0\| \leq r_0, \quad \forall t \in K,$$

which implies that $\mathbf{g} \in X$. □

Lemma 6.5. *Define the map* $F : X \to C(K; \mathbb{R}^d)$

$$[F(\mathbf{g})](t) := \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{g}(s))ds, \quad \forall t \in K,$$

where $\mathbf{f}$ *obeys (6.5). Then (i) the range of* F *belongs to* X *and (ii)* $F : X \to X$ *is a contraction.*

Proof.

(i). Since f_j are continuous real valued functions, we have that

$$K \ni s \mapsto f_j(s, \mathbf{g}(s)) \in \mathbb{R}$$

are also continuous, thus Riemann integrable. Because $\mathbf{g}(s) \in \overline{B_{r_0}(\mathbf{y}_0)}$ for all $s \in K$, we have that $(s, \mathbf{g}(s)) \in H_0$. The integral from the definition of F defines a vector $\mathbf{u}(t)$ with components

$$u_j(t) := \int_{t_0}^{t} f_j(s, \mathbf{g}(s))ds, \quad 1 \leq j \leq d.$$

Denote by $t_1 := \min\{t_0, t\}$ and $t_2 := \max\{t_0, t\}$. Then we have:

$$||\mathbf{u}(t)||^2 = \sum_{j=1}^{d} u_j^2(t) = \int_{t_0}^{t} \left(\sum_{j=1}^{d} u_j(t) f_j(s, \mathbf{g}(s)) \right) ds \leq \int_{t_1}^{t_2} ||\mathbf{u}(t)|| \, ||\mathbf{f}(s, \mathbf{g}(s))|| ds$$

where in the last inequality we used the Cauchy-Schwarz inequality. Hence we may write:

$$\left\| \int_{t_0}^{t} \mathbf{f}(s, \mathbf{g}(s))ds \right\| \leq \int_{t_1}^{t_2} ||\mathbf{f}(s, \mathbf{g}(s))|| ds.$$

From (6.6) we have $\sup_{s \in K} \|\mathbf{f}(s, \mathbf{g}(s))\| \leq M$, hence:

$$\|[F(\mathbf{g})](t) - \mathbf{y}_0\| = \|\mathbf{u}(t)\| \leq \int_{t_1}^{t_2} ||\mathbf{f}(s, \mathbf{g}(s))|| ds \leq M\delta < r_0, \quad \forall t \in K,$$

which shows that $[F(\mathbf{g})](t) \in \overline{B_{r_0}(\mathbf{y}_0)}$ for all $t \in K$, thus the range of F is contained in X.

(ii). Consider two functions $\mathbf{g}, \mathbf{h} \in X$. We have

$$d_\infty(F(\mathbf{g}), F(\mathbf{h})) = \sup_{t \in K} \|[F(\mathbf{g})](t) - [F(\mathbf{h})](t)\|.$$

The Lipschitz condition from (6.5) implies:

$$\begin{aligned} ||F(\mathbf{g})](t) - [F(\mathbf{h})](t)|| = \left\| \int_{t_0}^t [\mathbf{f}(s, \mathbf{g}(s)) - \mathbf{f}(s, \mathbf{h}(s))] ds \right\| &\leq (\delta\ L) \sup_{s \in K} ||\mathbf{g}(s) - \mathbf{h}(s)|| \\ &\leq (\delta\ L) d_\infty(\mathbf{g}, \mathbf{h}), \quad \forall t \in K. \end{aligned} \tag{6.9}$$

It means that $d_\infty(F(\mathbf{g}), F(\mathbf{h})) \leq \delta\ L\ d_\infty(\mathbf{g}, \mathbf{h})$ for all $\mathbf{g}, \mathbf{h} \in X$, and $\delta L < 1$. Thus F is a contraction. □

Finishing the proof of Theorem 6.3. Vi have seen that F is a contraction on X. Then Theorem 5.2 implies that there exists a continuous function $\mathbf{y} : K \to \overline{B_{r_0}(\mathbf{y}_0)}$ such that

$$\mathbf{y}(t) = [F(\mathbf{y})](t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) ds, \quad t \in [t_0 - \delta, t_0 + \delta].$$

But the right hand side is differentiable for $t \in (t_0 - \delta, t_0 + \delta)$ due to the fundamental theorem of calculus, and moreover,

$$\frac{d}{dt} \left(\int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) ds \right) = \mathbf{f}(t, \mathbf{y}(t)).$$

Thus (6.7) is satisfied. Finally, let us prove uniqueness. Assume that there exists another solution $\mathbf{z}$ obeying the conditions of the theorem. We have $\mathbf{z}(t_0) = \mathbf{y}_0$ and $\mathbf{z}$ is continuous because it is differentiable; moreover, $\mathbf{z}'$ is also continuous because it equals $\mathbf{f}(s, \mathbf{z}(s))$, and $\mathbf{z} \in X$. Thus applying once again the fundamental theorem of calculus we obtain:

$$\mathbf{z}(t) = \mathbf{z}(t_0) + \int_{t_0}^t \mathbf{z}'(s) ds = [F(\mathbf{z})](t), \quad \mathbf{z} = F(\mathbf{z}), \quad \mathbf{z} \in X.$$

Since F has a unique fixed point, we must have $\mathbf{z} = \mathbf{y}$. □

Remark 6.6. *Choose $0 < \delta < \delta_1$. Define the sequence of functions $\mathbf{y}_k : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^d$, $k \geq 1$, where $\mathbf{y}_1(t) = \mathbf{y}_0$ and*

$$\mathbf{y}_{k+1}(t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}_k(s)) ds, \quad k \geq 1.$$

We see that $\mathbf{y}_{k+1} = F(\mathbf{y}_k)$, where F is given by Lemma 6.5. A direct use of Lemma 6.5 (ii) implies that $\{\mathbf{y}_k\}_{k \geq 1}$ converges uniformly on the interval $[t_0 - \delta, t_0 + \delta]$ towards a continuous function $\mathbf{y}$ which obeys the fixed point equation

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_{t_0}^t \mathbf{f}(s, \mathbf{y}(s)) ds,$$

thus solving (6.7). This is Picard's iteration method.

7 The implicit function theorem

The Euclidean space $\mathbb{R}^m$ has a norm defined by $||\mathbf{x}|| = \sqrt{\sum_{j=1}^m |x_j|^2}$.

Lemma 7.1. *Let A be a $m \times n$ matrix with real components $\{a_{jk}\}$. Define the quantity $||A||_{\mathrm{HS}} := \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{n}|a_{jk}|^2}$. Then*

$$||A\mathbf{u}||_{\mathbb{R}^m} \leq ||A||_{\mathrm{HS}}\, ||\mathbf{u}||_{\mathbb{R}^n}, \qquad \forall \mathbf{u} \in \mathbb{R}^n. \tag{7.1}$$

Proof. From the Cauchy-Schwarz inequality we have:

$$|(A\mathbf{u})_j|^2 = \left(\sum_{k=1}^{n} a_{jk}u_k\right)^2 \leq \left(\sum_{k=1}^{n}|a_{jk}|^2\right)\sum_{k=1}^{n}|u_k|^2 = \sum_{k=1}^{n}|a_{jk}|^2||\mathbf{u}||_{\mathbb{R}^n}^2,$$

and the lemma follows after summation with respect to j. □

Lemma 7.2. *Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set and $K_\delta := \overline{B_\delta(\mathbf{x}_0)} = \{\mathbf{x} \in \mathbb{R}^d : \ ||\mathbf{x}-\mathbf{x}_0|| \leq \delta\}$ be a closed ball included in $\mathcal{O}$. Let $\phi : \mathcal{O} \mapsto \mathbb{R}$ be a $C^1(K_\delta)$ map (which means that $\partial_j\phi$ exist for all j and are continuous functions on K_δ). Denote by $||\partial_j\phi||_\infty = \sup_{\mathbf{x}\in K_\delta}|\partial_j\phi(\mathbf{x})| < \infty$. Then for every $\mathbf{x}, \mathbf{x}' \in K_\delta$ we have:*

$$|\phi(\mathbf{x}) - \phi(\mathbf{x}')| \leq \sqrt{\sum_{j=1}^{d}||\partial_j\phi||_\infty^2}\ ||\mathbf{x}-\mathbf{x}'||. \tag{7.2}$$

Proof. Define the real valued map $f(t) = \phi((1-t)\mathbf{x}' + t\mathbf{x})$, $0 \leq t \leq 1$. Applying the chain rule we obtain:

$$f'(t) = \sum_{j=1}^{d}(x_j - x_j')(\partial_j\phi)((1-t)\mathbf{x}' + t\mathbf{x}),$$

thus the Cauchy-Schwarz inequality implies:

$$|f'(t)| \leq \sqrt{\sum_{j=1}^{d}|\partial_j\phi((1-t)\mathbf{x}' + t\mathbf{x})|^2}\ ||\mathbf{x}-\mathbf{x}'|| \leq \sqrt{\sum_{j=1}^{d}||\partial_j\phi||_\infty^2}\ ||\mathbf{x}-\mathbf{x}'||, \quad \forall 0 < t < 1.$$

Since $\phi(\mathbf{x}) - \phi(\mathbf{x}') = f(1) - f(0) = \int_0^1 f'(t)dt$, we obtain:

$$|\phi(\mathbf{x}) - \phi(\mathbf{x}')| \leq \int_0^1 |f'(t)|dt \leq \sqrt{\sum_{j=1}^{d}||\partial_j\phi||_\infty^2}\ ||\mathbf{x}-\mathbf{x}'||$$

which proves (7.2). □

Lemma 7.3. *Let $\mathcal{O}$ and K_δ be as above. Let $\mathbf{f} : \mathcal{O} \mapsto \mathbb{R}^q$ a vector valued map which is $C^1(K_\delta)$. Define*

$$||\Delta\mathbf{f}||_{\infty,K_\delta} := \sqrt{\sum_{k=1}^{q}\sum_{j=1}^{d}||\partial_j f_k||_\infty^2}.$$

Then we have:

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}')||_{\mathbb{R}^q} \leq ||\Delta\mathbf{f}||_{\infty,K_\delta}||\mathbf{x}-\mathbf{x}'||_{\mathbb{R}^d}, \quad \forall \mathbf{x}, \mathbf{x}' \in K_\delta. \tag{7.3}$$

Proof. Let $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), ..., f_q(\mathbf{x})]$ and use (7.2) with ϕ replaced by f_k, $1 \leq k \leq q$. We have:

$$|f_k(\mathbf{x}) - f_k(\mathbf{x}')|^2 \leq \sum_{j=1}^{d}||\partial_j f_k||_\infty^2||\mathbf{x}-\mathbf{x}'||_{\mathbb{R}^d}^2$$

and the proof is completed after taking the sum over k. □

Let $d = m + n$ with $1 \le m, n < d$. A vector $\mathbf{x} \in \mathbb{R}^d$ can be uniquely decomposed as $\mathbf{x} = [\mathbf{u}, \mathbf{w}]$ with $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^n$. Let $U \in \mathbb{R}^d$ be an open set and $\mathbf{h} : U \mapsto \mathbb{R}^m$ be a $C^1(U; \mathbb{R}^m)$ function. We denote by:

$$[D_{\mathbf{u}}\mathbf{h}([\mathbf{u}', \mathbf{w}'])] := \left\{ \frac{\partial h_k}{\partial u_j}([\mathbf{u}', \mathbf{w}']) : \ 1 \le j, k \le m \right\} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m),$$

and

$$[D_{\mathbf{w}}\mathbf{h}([\mathbf{u}', \mathbf{w}'])] := \left\{ \frac{\partial h_k}{\partial w_j}([\mathbf{u}', \mathbf{w}']) : \ 1 \le k \le m, \ 1 \le j \le n \right\} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

the partial Jacobi matrices of $\mathbf{h}$. We also have:

$$[D\mathbf{h}([\mathbf{u}', \mathbf{w}'])] = [D_{\mathbf{u}}\mathbf{h}([\mathbf{u}', \mathbf{w}']); D_{\mathbf{w}}\mathbf{h}([\mathbf{u}', \mathbf{w}'])] \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m).$$

We can now formulate the implicit function theorem.

Theorem 7.4. *Let $U \subset \mathbb{R}^d$ be an open set and $\mathbf{h} : U \mapsto \mathbb{R}^m$ be a $C^1(U; \mathbb{R}^m)$ function. Assume that there exists a point $\mathbf{a} = [\mathbf{u_a}, \mathbf{w_a}] \in U$ such that $\mathbf{h}(\mathbf{a}) = 0$ and the $m \times m$ partial Jacobi matrix $[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]$ is invertible. Then there exists an open set $E \subset \mathbb{R}^n$ containing $\mathbf{w_a}$ and a map $\mathbf{f} : E \mapsto \mathbb{R}^m$ which obeys $\mathbf{f}(\mathbf{w_a}) = \mathbf{u_a}$ and $\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}]) = 0$ for all $\mathbf{w} \in E$. Moreover, the matrix $[D_{\mathbf{u}}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])]$ is invertible if $\mathbf{w} \in E$ and all entries of its inverse are continuous on E. Finally, $\mathbf{f}$ is continuously differentiable on E and we have:*

$$[D\mathbf{f}(\mathbf{w})] = -[D_{\mathbf{u}}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])]^{-1} \, [D_{\mathbf{w}}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), \quad \forall \mathbf{w} \in E. \tag{7.4}$$

Proof. The point $\mathbf{a}$ is an interior point of U, hence there exists $r > 0$ such that $B_r(\mathbf{a}) \subset U$. Thus for every $\mathbf{x} = [\mathbf{u}, \mathbf{w}] \in B_r(\mathbf{a})$ we have

$$||\mathbf{x} - \mathbf{a}||^2_{\mathbb{R}^d} = ||\mathbf{u} - \mathbf{u_a}||^2_{\mathbb{R}^m} + ||\mathbf{w} - \mathbf{w_a}||^2_{\mathbb{R}^n} < r^2.$$

If $\epsilon < r/\sqrt{2}$, let $P_n(\epsilon)$ be the open ball $B_\epsilon(\mathbf{w_a}) \subset \mathbb{R}^n$ and $Q_m(\epsilon)$ be the open ball $B_\epsilon(\mathbf{u_a}) \subset \mathbb{R}^m$. Then one can verify that $\overline{Q_m(\epsilon)} \times P_n(\epsilon) \subset B_r(\mathbf{a}) \subset U$.

For every $\mathbf{w} \in P_n(\epsilon)$, define the map $F_{\mathbf{w}} : \overline{Q_m(\epsilon)} \mapsto \mathbb{R}^m$ given by

$$F_{\mathbf{w}}(\mathbf{u}) := \mathbf{u} - [D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}\mathbf{h}([\mathbf{u}, \mathbf{w}]).$$

The main idea behind the construction is to show that there exists a constant $C > 0$ and some small some enough $\epsilon_1 < r/\sqrt{2}$ such that for every $\epsilon \le \epsilon_1$ and for every $||\mathbf{w} - \mathbf{w_a}||_{\mathbb{R}^n} < \epsilon/(10C)$ the following two facts hold true:

1. $F_{\mathbf{w}}\left(\overline{Q_m(\epsilon)}\right) \subset \overline{Q_m(\epsilon)}$ and

2. $F_{\mathbf{w}} : \overline{Q_m(\epsilon)} \mapsto \overline{Q_m(\epsilon)}$ is a contraction.

Then Banach's Fixed Point Theorem provides us with a unique $\mathbf{u_w} \in \overline{Q_m(\epsilon)}$ such that $F_{\mathbf{w}}(\mathbf{u_w}) = \mathbf{u_w}$. This would imply that $[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}\mathbf{h}([\mathbf{u_w}, \mathbf{w}]) = 0$; multiplying with $[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]$ on the left leads to $\mathbf{h}([\mathbf{u_w}, \mathbf{w}]) = 0$. This is how we construct the map $\mathbf{f}(\mathbf{w}) := \mathbf{u_w}$ for all $\mathbf{w}$ obeying $||\mathbf{w} - \mathbf{w_a}||_{\mathbb{R}^n} < \epsilon_1/(10C)$. In the second part of the proof one needs to show that $\mathbf{f}$ is also continuously differentiable when restricted to a ball around $\mathbf{w_a}$ and obeys (7.4). Technical details are given below.

Step 1: constructing $\mathbf{f}(\mathbf{w})$ as a fixed point.

Let us start with an estimate which will play an important role in what follows. We want to prove that there exists some $0 < \epsilon_1 < r/\sqrt{2}$ small enough such that for every $\epsilon \le \epsilon_1$ and $\mathbf{w} \in P_n(\epsilon)$ we have:

$$||F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}}(\mathbf{u}')||_{\mathbb{R}^m} \le \frac{1}{10}||\mathbf{u} - \mathbf{u}'||_{\mathbb{R}^m}, \quad \forall \mathbf{u}, \mathbf{u}' \in \overline{Q_m(\epsilon)}. \tag{7.5}$$

From the definition of $F_{\mathbf{w}}$ we can write:

$$F_{\mathbf{w}}(\mathbf{u}) = -[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}\{\mathbf{h}([\mathbf{u},\mathbf{w}]) - [D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]\mathbf{u}\}.$$

Define $\mathbf{g}_{\mathbf{w}}(\mathbf{u}) := \mathbf{h}([\mathbf{u},\mathbf{w}]) - [D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]\mathbf{u}$. Hence $F_{\mathbf{w}}(\mathbf{u}) = -[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}\mathbf{g}_{\mathbf{w}}(\mathbf{u})$ and we have:

$$||F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}}(\mathbf{u}')||_{\mathbb{R}^m} \leq ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}||_{\mathrm{HS}}||\mathbf{g}_{\mathbf{w}}(\mathbf{u}) - \mathbf{g}_{\mathbf{w}}(\mathbf{u}')||_{\mathbb{R}^m}, \tag{7.6}$$

where we used (7.1).

The set $\overline{Q_m(\epsilon)}$ is closed and bounded, thus compact. As in Lemma 7.3, namely the estimate (7.3) with $q = m = d$, we can derive the inequality:

$$||\mathbf{g}_{\mathbf{w}}(\mathbf{u}) - \mathbf{g}_{\mathbf{w}}(\mathbf{u}')||_{\mathbb{R}^m} \leq ||\Delta\mathbf{g}_{\mathbf{w}}||_{\infty,\overline{Q_m(\epsilon)}}\; ||\mathbf{u} - \mathbf{u}'||_{\mathbb{R}^m}, \quad \forall \mathbf{u},\mathbf{u}' \in \overline{Q_m(\epsilon)}, \quad \mathbf{w} \in P_n(\epsilon). \tag{7.7}$$

Let us show that $||\Delta\mathbf{g}_{\mathbf{w}}||_{\infty,\overline{Q_m(\epsilon)}}$ can be made arbitrarily small when ϵ goes to zero. By computing the partial derivatives we obtain:

$$\frac{\partial[\mathbf{g}_{\mathbf{w}}]_k}{\partial u_j}(\mathbf{u}) = \frac{\partial h_k}{\partial u_j}([\mathbf{u},\mathbf{w}]) - \frac{\partial h_k}{\partial u_j}(\mathbf{a}).$$

Due to the continuity of the partial derivatives of $\mathbf{h}$ at $\mathbf{a}$, we get that the above right hand side can be made arbitrarily small with ϵ. In particular, there exists $\epsilon_1 > 0$ small enough such that

$$||\Delta\mathbf{g}_{\mathbf{w}}||_{\infty,\overline{Q_m(\epsilon_1)}} \leq \frac{1}{10(1 + ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}||_{\mathrm{HS}}||)}, \quad \forall \mathbf{w} \in P_n(\epsilon_1).$$

Inserting this in (7.7) and then in (7.6), we prove (7.5).

We need a second important estimate. We will show that there exists a constant $C \geq 1$ such that

$$||F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}'}(\mathbf{u})||_{\mathbb{R}^m} \leq C\; ||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}, \quad \forall \mathbf{w},\mathbf{w}' \in P_n(\epsilon_1), \quad \forall \mathbf{u} \in \overline{Q_m(\epsilon_1)}. \tag{7.8}$$

Indeed, using the definition we have:

$$F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}'}(\mathbf{u}) = -[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}\{\mathbf{h}([\mathbf{u},\mathbf{w}]) - \mathbf{h}([\mathbf{u},\mathbf{w}'])\}.$$

Now reasoning as in Lemma 7.3 in which we put $\mathcal{O} = U$, $K_\delta = \overline{Q_m(\epsilon_1)} \times \overline{P_n(\epsilon_1)}$, $q = m$, $d = m+n$, $\mathbf{x} = [\mathbf{u},\mathbf{w}]$ and $\mathbf{x}' = [\mathbf{u},\mathbf{w}']$ we obtain

$$||\mathbf{h}([\mathbf{u},\mathbf{w}]) - \mathbf{h}([\mathbf{u},\mathbf{w}'])||_{\mathbb{R}^m} \leq ||\Delta\mathbf{h}||_{\infty,K_\delta}\; ||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}.$$

Thus

$$||F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}'}(\mathbf{u})||_{\mathbb{R}^m} \leq ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}||_{\mathrm{HS}}||\Delta\mathbf{h}||_{\infty,K_\delta}||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}$$

and we can choose $C := 1 + ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]^{-1}||_{\mathrm{HS}}||\Delta\mathbf{h}||_{\infty,K_\delta}$. Hence we obtain (7.8).

In particular, if $\mathbf{w}' = \mathbf{w}_{\mathbf{a}}$ and $\mathbf{u} = \mathbf{u}_{\mathbf{a}}$ we obtain:

$$||F_{\mathbf{w}}(\mathbf{u}_{\mathbf{a}}) - F_{\mathbf{w}_{\mathbf{a}}}(\mathbf{u}_{\mathbf{a}})||_{\mathbb{R}^m} \leq C\; ||\mathbf{w} - \mathbf{w}_{\mathbf{a}}||_{\mathbb{R}^n}, \quad \forall \mathbf{w} \in P_n(\epsilon_1). \tag{7.9}$$

This implies:

$$||F_{\mathbf{w}}(\mathbf{u}_{\mathbf{a}}) - F_{\mathbf{w}_{\mathbf{a}}}(\mathbf{u}_{\mathbf{a}})||_{\mathbb{R}^m} \leq \frac{\epsilon}{10}, \quad \forall \mathbf{w} \in P_n(\epsilon/(10C)), \quad \epsilon \leq \epsilon_1. \tag{7.10}$$

We are now able to prove that for every $\epsilon \leq \epsilon_1$ and for every $\mathbf{w} \in P_n(\epsilon/(10C))$, the map $F_{\mathbf{w}}$ leaves the set $\overline{Q_m(\epsilon)}$ invariant. Note first that $F_{\mathbf{w}_{\mathbf{a}}}(\mathbf{u}_{\mathbf{a}}) = \mathbf{u}_{\mathbf{a}}$ because $\mathbf{h}(\mathbf{a}) = \mathbf{0}$ from our hypothesis. Now if $||\mathbf{u} - \mathbf{u}_{\mathbf{a}}|| \leq \epsilon \leq \epsilon_1$ we have:

$$||F_{\mathbf{w}}(\mathbf{u}) - \mathbf{u}_{\mathbf{a}}||_{\mathbb{R}^m} \leq ||F_{\mathbf{w}}(\mathbf{u}) - F_{\mathbf{w}}(\mathbf{u}_{\mathbf{a}})||_{\mathbb{R}^m} + ||F_{\mathbf{w}}(\mathbf{u}_{\mathbf{a}}) - \mathbf{u}_{\mathbf{a}}||_{\mathbb{R}^m} \leq \frac{\epsilon}{5},$$

where we used both (7.5) and (7.10). In particular, $F_{\mathbf{w}}(\mathbf{u}) \in \overline{Q_m(\epsilon)}$.

We have just proved that for every $\epsilon \le \epsilon_1$ and $\mathbf{w} \in P_n(\epsilon/(10C))$, the map $F_{\mathbf{w}} : \overline{Q_m(\epsilon)} \mapsto \overline{Q_m(\epsilon)}$ is a contraction (see (7.5)) defined on the complete metric space $\overline{Q_m(\epsilon)} \subset \mathbb{R}^m$. Thus there exists a unique $\mathbf{u_w} \in \overline{Q_m(\epsilon)}$ such that $F_{\mathbf{w}}(\mathbf{u_w}) = \mathbf{u_w}$, which implies that

$$\mathbf{h}([\mathbf{u_w}, \mathbf{w}]) = 0, \quad \forall \mathbf{w} \in P_n(\epsilon/(10C)).$$

Step 2: f is continuous on $P_n(\epsilon_1/(10C))$.

If $\mathbf{w}, \mathbf{w}' \in P_n(\epsilon_1/(10C))$ we have:

$$\begin{aligned} ||\mathbf{u_w} - \mathbf{u_{w'}}||_{\mathbb{R}^m} &= ||F_{\mathbf{w}}(\mathbf{u_w}) - F_{\mathbf{w}'}(\mathbf{u_{w'}})||_{\mathbb{R}^m} \\ &\le ||F_{\mathbf{w}}(\mathbf{u_w}) - F_{\mathbf{w}}(\mathbf{u_{w'}})||_{\mathbb{R}^m} + ||F_{\mathbf{w}}(\mathbf{u_{w'}}) - F_{\mathbf{w}'}(\mathbf{u_{w'}})||_{\mathbb{R}^m} \\ &\le \frac{1}{10} ||\mathbf{u_w} - \mathbf{u_{w'}}||_{\mathbb{R}^m} + C\, ||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}. \end{aligned} \tag{7.11}$$

This shows that

$$||\mathbf{u_w} - \mathbf{u_{w'}}||_{\mathbb{R}^m} \le \frac{10C}{9} ||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}, \quad \forall \mathbf{w}, \mathbf{w}' \in P_n(\epsilon_1/(10C)), \tag{7.12}$$

which proves that the map

$$P_n(\epsilon_1/(10C)) \ni \mathbf{w} \mapsto \mathbf{u_w} =: \mathbf{f}(\mathbf{w}) \in \mathbb{R}^m$$

is (Lipschitz) continuous.

Step 3: f is differentiable.

We now want to prove that there exists some $\epsilon_2 \le \epsilon_1$ such that $\mathbf{f}(\mathbf{w}) = \mathbf{u_w}$ is differentiable on $E := P_n(\epsilon_2/(10C))$ and obeys (7.4). Because $\mathbf{h}$ is differentiable at $\mathbf{x}' = [\mathbf{u}', \mathbf{w}']$, there exists a map $\varepsilon_{\mathbf{x}'}$ defined on the ball $B_r(\mathbf{a}) \subset \mathbb{R}^d$, continuous at $\mathbf{x}'$ and with $\varepsilon_{\mathbf{x}'}(\mathbf{x}') = 0$, such that for every $\mathbf{x} \in B_r(\mathbf{a})$ we can write:

$$\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}') = [D\mathbf{h}(\mathbf{x}')](\mathbf{x} - \mathbf{x}') + ||\mathbf{x} - \mathbf{x}'||_{\mathbb{R}^d} \varepsilon_{\mathbf{x}'}(\mathbf{x}). \tag{7.13}$$

Replacing $\mathbf{x}$ with $[\mathbf{f}(\mathbf{w}), \mathbf{w}]$ and $\mathbf{x}'$ with $[\mathbf{f}(\mathbf{w}'), \mathbf{w}']$ we have:

$$\begin{aligned} &\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}]) - \mathbf{h}([\mathbf{f}(\mathbf{w}'), \mathbf{w}']) \\ &= [D\mathbf{h}(\mathbf{x}')]([\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}'), \mathbf{w} - \mathbf{w}']) + \sqrt{||\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}')||^2_{\mathbb{R}^m} + ||\mathbf{w} - \mathbf{w}'||^2_{\mathbb{R}^n}}\; \varepsilon_{\mathbf{x}'}([\mathbf{f}(\mathbf{w}), \mathbf{w}]). \end{aligned} \tag{7.14}$$

Because $\mathbf{h}$ is a C^1 function, the map

$$\overline{Q_m(\epsilon)} \times P_n(\epsilon/(10C)) \ni \mathbf{x}' \mapsto \det[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')] \in \mathbb{R}, \quad \epsilon \le \epsilon_1$$

is continuous. Since $[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})]$ is invertible we must have that $\det[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})] \ne 0$. When the point $\mathbf{x}'$ belongs to $\overline{Q_m(\epsilon)} \times P_n(\epsilon/(10C))$, by choosing $\epsilon =: \epsilon_3$ small enough we can insure that $\det[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')] \ne 0$ for all $\mathbf{x}' \in \overline{Q_m(\epsilon_3)} \times P_n(\epsilon_3/(10C))$, thus $[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]$ is invertible.

Remember the identities $\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}]) = \mathbf{h}([\mathbf{f}(\mathbf{w}'), \mathbf{w}']) = 0$ and

$$[D\mathbf{h}(\mathbf{x}')](\mathbf{x} - \mathbf{x}') = [D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')](\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}')) + [D_{\mathbf{w}}\mathbf{h}(\mathbf{x}')](\mathbf{w} - \mathbf{w}').$$

Using these facts in (7.14) and multiplying both sides of (7.14) with $[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}$ we obtain:

$$\begin{aligned} \mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}') =& - [D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}[D_{\mathbf{w}}\mathbf{h}(\mathbf{x}')](\mathbf{w} - \mathbf{w_a}) \\ &- \sqrt{||\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}')||^2_{\mathbb{R}^m} + ||\mathbf{w} - \mathbf{w}'||^2_{\mathbb{R}^n}}\; [D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}\varepsilon_{\mathbf{x}'}([\mathbf{f}(\mathbf{w}), \mathbf{w}]). \end{aligned} \tag{7.15}$$

Using (7.12) we have:

$$\sqrt{||\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}')||^2_{\mathbb{R}^m} + ||\mathbf{w} - \mathbf{w}'||^2_{\mathbb{R}^n}} \le \sqrt{\frac{100C^2}{81} + 1}\; ||\mathbf{w} - \mathbf{w}'||_{\mathbb{R}^n}, \quad \forall \mathbf{w}, \mathbf{w}' \in P_n(\epsilon_3/(10C)).$$

Replacing this in (7.15) we obtain:

$$
\begin{aligned}
&||\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}') + [D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}[D_{\mathbf{w}}\mathbf{h}(\mathbf{x}')](\mathbf{w}-\mathbf{w}')||_{\mathbb{R}^m} \\
&\leq \sqrt{\frac{100C^2}{81}+1}\, ||\mathbf{w}-\mathbf{w}'||_{\mathbb{R}^n} ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}\varepsilon_{\mathbf{x}'}([\mathbf{f}(\mathbf{w}),\mathbf{w}])||_{\mathbb{R}^m} \\
&\leq ||\mathbf{w}-\mathbf{w}'||_{\mathbb{R}^n}\sqrt{\frac{100C^2}{81}+1}\, ||[D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}||_{\mathrm{HS}} ||\varepsilon_{\mathbf{x}'}([\mathbf{f}(\mathbf{w}),\mathbf{w}])||_{\mathbb{R}^m}, \quad \forall \mathbf{w}\in P_n(\epsilon_3/(10C)). \quad (7.16)
\end{aligned}
$$

Now using $\lim_{\mathbf{w}\to\mathbf{w}'}[\mathbf{f}(\mathbf{w}),\mathbf{w}] = \mathbf{x}'$ and the continuity of $\varepsilon_{\mathbf{x}'}$ at $\mathbf{x} = \mathbf{x}'$ we have:

$$
\lim_{\mathbf{w}\to\mathbf{w}'} \frac{||\mathbf{f}(\mathbf{w}) - \mathbf{f}(\mathbf{w}') + [D_{\mathbf{u}}\mathbf{h}(\mathbf{x}')]^{-1}[D_{\mathbf{w}}\mathbf{h}(\mathbf{x}')](\mathbf{w}-\mathbf{w}')||_{\mathbb{R}^m}}{||\mathbf{w}-\mathbf{w}'||_{\mathbb{R}^n}} = 0,
$$

which shows that $\mathbf{f}$ is differentiable at $\mathbf{w}'$ and proves (7.4). Moreover, because the right hand side of (7.4) is continuous on $E = P_n(\epsilon_3/(10C))$ it follows that $\mathbf{f}\in C^1(E;\mathbb{R}^m)$ and we are done. □

8 The inverse function theorem

We start with two technical lemmas.

Lemma 8.1. *Let $\mathcal{O}\subset\mathbb{R}^m$ be an open set, $K_\delta := \overline{B_\delta(\mathbf{u}_0)} = \{\mathbf{u}\in\mathbb{R}^m :\ ||\mathbf{u}-\mathbf{u}_0||\leq\delta\}$ be a closed ball included in $\mathcal{O}$, and $\mathbf{f}:\mathcal{O}\mapsto\mathbb{R}^m$ such that $\mathbf{f}\in C^1(K_\delta)$. Define $\mathbf{g}(\mathbf{u}) = \mathbf{f}(\mathbf{u}) - [D\mathbf{f}(\mathbf{u}_0)]\mathbf{u}$ on K_δ, where $[D\mathbf{f}(\mathbf{u}_0)]$ is the Jacobi matrix with elements $[D\mathbf{f}(\mathbf{u}_0)]_{kj} = (\partial_j f_k)(\mathbf{u}_0)$. Then for every $\beta>0$ there exists an $0<\epsilon_\beta<\delta$ such that for every $0<\epsilon<\epsilon_\beta$ we have:*

$$
||\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{u}')|| \leq \beta||\mathbf{u}-\mathbf{u}'||, \quad \forall \mathbf{u},\mathbf{u}'\in K_\epsilon. \quad (8.1)
$$

Proof. A straightforward computation gives $\partial_j g_k(\mathbf{x}) = \partial_j f_k(\mathbf{x}) - \partial_j f_k(\mathbf{u}_0)$. Thus $||\partial_j g_k||_\infty$ can be made arbitrarily small when ϵ gets smaller, because $\mathbf{f}$ has continuous partial derivatives. It follows that $||\Delta\mathbf{g}||_{\infty,K_\epsilon}\leq\beta$ whenever ϵ gets smaller than some small enough ϵ_β, and then we can use (7.3) with $\mathbf{g}$ instead of $\mathbf{f}$. □

Lemma 8.2. *Let $\mathcal{O}\subset\mathbb{R}^m$ be open and let $\mathbf{u}_0\in\mathcal{O}$. Let $\mathbf{f}$ be a $C^1(\mathcal{O};\mathbb{R}^m)$ vector valued function, such that $[D\mathbf{f}(\mathbf{u}_0)]\in\mathcal{L}(\mathbb{R}^m;\mathbb{R}^m)$ is an invertible matrix. Then there exists $r>0$ small enough such that the restriction of $\mathbf{f}$ to $B_r(\mathbf{u}_0)$ is injective.*

Proof. Assume the contrary: for every $n\geq1$ there exist two different points $\mathbf{x}_n\neq\mathbf{y}_n$ in $B_{\frac{1}{n}}(\mathbf{u}_0)$ such that $\mathbf{f}(\mathbf{x}_n) = \mathbf{f}(\mathbf{y}_n)$. Define $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - [D\mathbf{f}(\mathbf{u}_0)]\mathbf{x}$ on $B_{\frac{1}{n}}(\mathbf{u}_0)$. Then we have $\mathbf{g}(\mathbf{x}_n) - \mathbf{g}(\mathbf{y}_n) = [D\mathbf{f}(\mathbf{u}_0)](\mathbf{y}_n-\mathbf{x}_n)$ or:

$$
\mathbf{y}_n - \mathbf{x}_n = [D\mathbf{f}(\mathbf{u}_0)]^{-1}(\mathbf{g}(\mathbf{x}_n) - \mathbf{g}(\mathbf{y}_n)), \quad \forall n\geq1.
$$

Now using (7.1) we have:

$$
||\mathbf{y}_n - \mathbf{x}_n|| = ||[D\mathbf{f}(\mathbf{u}_0)]^{-1}||_{\mathrm{HS}}\, ||\mathbf{g}(\mathbf{x}_n) - \mathbf{g}(\mathbf{y}_n)||, \quad \forall n\geq1.
$$

Choosing $\beta = \frac{1}{1+||[D\mathbf{f}(\mathbf{u}_0)]^{-1}||_{\mathrm{HS}}}$, then from (8.1) we infer that there exists some $\epsilon_\beta>0$ sufficiently small such that for every $n^{-1}<\epsilon_\beta$ we have $||\mathbf{g}(\mathbf{x}_n) - \mathbf{g}(\mathbf{y}_n)||\leq\beta||\mathbf{y}_n-\mathbf{x}_n||$. It follows that:

$$
||\mathbf{y}_n - \mathbf{x}_n|| \leq \frac{||[D\mathbf{f}(\mathbf{u}_0)]^{-1}||_{\mathrm{HS}}}{1+||[D\mathbf{f}(\mathbf{u}_0)]^{-1}||_{\mathrm{HS}}}||\mathbf{y}_n-\mathbf{x}_n|| < ||\mathbf{y}_n-\mathbf{x}_n||, \quad \forall 0<n^{-1}<\epsilon_\beta,
$$

which contradicts the assumption $||\mathbf{y}_n-\mathbf{x}_n||\neq0$. □

Here is the Inverse Function Theorem:

Theorem 8.3. *Let $\mathcal{O} \subset \mathbb{R}^m$ be an open set containing $\mathbf{u}_0$. Let $\mathbf{g} \in C^1(\mathcal{O};\mathbb{R}^m)$ such that $[D\mathbf{g}(\mathbf{u}_0)] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is invertible, and $\mathbf{g}$ is injective on $\mathcal{O}$. Then there exists an open ball $E \subset \mathbb{R}^m$ which contains $\mathbf{w}_0 := \mathbf{g}(\mathbf{u}_0)$, and a function $\mathbf{f} : E \mapsto \mathcal{O}$ such that the following facts hold true:*

(i). The set $V = \mathbf{f}(E)$ equals $\mathbf{g}^{-1}(E)$ and is open in $\mathbb{R}^m$;

(ii). $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w}$ on E and $\mathbf{f}(\mathbf{g}(\mathbf{u})) = \mathbf{u}$ on V, hence they are local inverses to each other;

(iii). The function $\mathbf{f}$ is a $C^1(V)$ function, $[D\mathbf{g}(\mathbf{f}(\mathbf{w}))]$ is invertible on E and we have:

$$[D\mathbf{f}(\mathbf{w})] = [D\mathbf{g}(\mathbf{f}(\mathbf{w}))]^{-1}.$$

Proof. The set $U := \mathcal{O} \times \mathbb{R}^m \subset \mathbb{R}^{2m}$ is open. Define $\mathbf{h} : U \mapsto \mathbb{R}^m$ given by $\mathbf{h}([\mathbf{u}, \mathbf{w}]) := \mathbf{g}(\mathbf{u}) - \mathbf{w}$. Denote by $\mathbf{a} := [\mathbf{u}_0, \mathbf{w}_0]$. Then $\mathbf{h} \in C^1(U)$, $\mathbf{h}(\mathbf{a}) = 0$, and $[D_{\mathbf{u}}\mathbf{h}(\mathbf{a})] = [D\mathbf{g}(\mathbf{u}_0)]$ is invertible. Thus the conditions of the Implicit Function Theorem are satisfied and we can find an open ball $E \subset \mathbb{R}^m$ containing $\mathbf{g}(\mathbf{u}_0) = \mathbf{w}_0$ and a function $\mathbf{f} \in C^1(E)$ such that $\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}]) = 0$ on E. In other words, $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w}$ for every $\mathbf{w} \in E$. This equality shows in particular that $\mathbf{g}(\mathbf{y}) \in E$ if $\mathbf{y}$ is of the form $\mathbf{f}(\mathbf{w})$ with $\mathbf{w} \in E$. In other words, $\mathbf{f}(E) \subset \mathbf{g}^{-1}(E)$.

Now let us show that in fact $\mathbf{f}(E) = \mathbf{g}^{-1}(E)$. Let $\mathbf{x} \in \mathbf{g}^{-1}(E)$. We have $\mathbf{g}(\mathbf{x}) =: \mathbf{w} \in E$ hence $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w} = \mathbf{g}(\mathbf{x})$. Because $\mathbf{g}$ is injective on $\mathcal{O}$ we must have $\mathbf{x} = \mathbf{f}(\mathbf{w}) \in \mathbf{f}(E)$, hence $\mathbf{g}^{-1}(E) \subset \mathbf{f}(E)$ and the equality of the two sets is proved.

Since $\mathbf{g}$ is continuous, the set $V = \mathbf{f}(E) = \mathbf{g}^{-1}(E)$ is open according to Theorem 4.2. This proves (i), together with the equality $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w}$ on E.

Now let us prove that we also have $\mathbf{f}(\mathbf{g}(\mathbf{u})) = \mathbf{u}$ on V. Take $\mathbf{u} \in V = \mathbf{g}^{-1}(E)$ and put $\mathbf{w} = \mathbf{g}(\mathbf{u}) \in E$. Since $\mathbf{w} = \mathbf{g}(\mathbf{f}(\mathbf{w}))$, we must have $\mathbf{g}(\mathbf{u}) = \mathbf{g}(\mathbf{f}(\mathbf{w}))$. Because $\mathbf{g}$ is injective, we must have $\mathbf{u} = \mathbf{f}(\mathbf{w}) = \mathbf{f}(\mathbf{g}(\mathbf{u})$, thus (ii) is proved.

Finally, differentiating $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w}$ and using the chain rule we get

$$[D\mathbf{g}(\mathbf{f}(\mathbf{w}))][D\mathbf{f}(\mathbf{w})] = I_{m\times m}$$

which means that both factors on the left hand side are invertible and (iii) is proved. □

9 Brouwer's fixed point theorem

We say that $K \subset \mathbb{R}^d$ is convex if for every $\mathbf{x}, \mathbf{y} \in K$ we have that $(1-t)\mathbf{x} + t\mathbf{y} \in K$ for all $0 \leq t \leq 1$. A set K is called a convex body if K is convex, compact, and with at least one interior point.

Theorem 9.1. *Let $K \subset \mathbb{R}^d$ be a convex body. Let $\mathbf{f} : K \mapsto K$ be a continuous function which invariates K. Then $\mathbf{f}$ has a (not necessarily unique) fixed point, that is a point $\mathbf{x} \in K$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof. The first thing we do is to reduce the problem from a general convex body to the unit ball in $\mathbb{R}^d$. We will show that there exists a bijection $\varphi : K \mapsto \overline{B_1(0)}$, which is continuous and with continuous inverse (a homeomorphism). If this is true, then it is enough to show that the function $\varphi \circ \mathbf{f} \circ \varphi^{-1} : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ has a fixed point $\mathbf{a} \in \overline{B_1(0)}$. In that case, $\mathbf{x} = \varphi^{-1}(\mathbf{a}) \in K$.

Lemma 9.2. *Any convex body in $\mathbb{R}^d$ is homeomorphic with the closed unit ball $\overline{B_1(0)}$.*

Proof. Let $\mathbf{x}_0$ be an interior point of K. There exists $r > 0$ such that $\overline{B_r(\mathbf{x}_0)} \subset K$. Define the continuous map $g : K \mapsto \mathbb{R}^d$ given by $g(\mathbf{x}) := (\mathbf{x} - \mathbf{x}_0)r^{-1}$. Define $\tilde{K} := g(K)$. It is easy to see that $\tilde{K}$ is a convex and compact set. Moreover, the function $g : K \mapsto \tilde{K}$ is invertible and $g^{-1}(\mathbf{y}) = r\mathbf{y} + \mathbf{x}_0$. Both g and g^{-1} are continuous, and for every $\mathbf{y} \in \mathbb{R}^d$ with $||\mathbf{y}|| \leq 1$ we have that $r\mathbf{y} + \mathbf{x}_0 \in \overline{B_r(\mathbf{x}_0)} \subset K$, thus $\mathbf{y} \in \tilde{K}$. This shows that $\overline{B_1(0)} \subset \tilde{K}$, thus $\tilde{K}$ is a convex body containing the closed unit ball.

We now construct a homeomorphism between $\tilde{K}$ and $\overline{B_1(0)}$. The boundary of $\tilde{K}$ is denoted by $\partial\tilde{K}$, equals $\tilde{K} \setminus \mathrm{int}(\tilde{K})$, and is a closed and bounded set. The boundary of $\overline{B_1(0)}$ is denoted by S^{d-1} and equals the unit sphere in $\mathbb{R}^d$. The ray connecting a given $\mathbf{x} \in \partial\tilde{K}$ with the origin intersects S^{d-1} in a point; in this way we define the function $h : \partial\tilde{K} \mapsto S^{d-1}$ where $h(\mathbf{x})$ is given by the above intersection.

The function h is injective; let us show this. Given $\mathbf{x} \in \partial\tilde{K}$, the set $C(\mathbf{x})$ defined by joining $\mathbf{x}$ with all the points of $\overline{B_1(0)}$ must belong to the convex set $\tilde{K}$. But $C(\mathbf{x})$ also contains the ray joining $\mathbf{x}$ with 0, and all the points of the segment strictly between $\mathbf{x}$ and 0 are interior points of $C(\mathbf{x})$, thus interior points of $\tilde{K}$. It means that no two different points of $\partial\tilde{K}$ can be placed on the same ray starting from the origin, which proves the injectivity of h.

Let us show that the function h is also surjective. Consider any ray generated by $\hat{x} \in S^{d-1}$, starting from the origin and parametrized by $R(\hat{x}) := \{\lambda\hat{x} : \ \lambda \geq 0\}$. This ray is a closed set. Consider the set $E(\hat{x}) := R(\hat{x}) \cap \mathrm{int}(\tilde{K})$. This set must be bounded, because $\tilde{K}$ is bounded. Hence the set of non-negative real numbers $\{||\mathbf{y}|| : \ \mathbf{y} \in E(\hat{x})\} \subset \mathbb{R}$ is bounded, thus it has a supremum $c < \infty$. The supremum is an accumulation point, thus there must exist a sequence of points $\{\mathbf{y}_n\}_{n\geq 1} \subset E(\hat{x})$ such that $||\mathbf{y}_n|| \to c$. But this sequence is also included in the compact set $\tilde{K} \cap R(\hat{x})$. It means that there exists a subsequence $\mathbf{y}_{n_k}$ which converges to some point $\mathbf{u} \in \tilde{K} \cap R(\hat{x})$, i.e. $||\mathbf{y}_{n_k} - \mathbf{u}|| \to 0$ when $k \to \infty$. Thus we also have $||\mathbf{y}_{n_k}|| \to ||\mathbf{u}||$ which shows that $||\mathbf{u}|| = c$. Now $\mathbf{u}$ cannot be an interior point of $\tilde{K}$, because in that case we could find points of $E(\hat{x})$ which are farther away from the origin than $\mathbf{u}$, contradicting the maximality of $||\mathbf{u}||$. Thus $\mathbf{u} \in \partial\tilde{K}$, which proves that the ray hits at least one point of the boundary.

Thus h is bijective and invertible. The (sequential) continuity of h follows easily by geometric arguments.

Let us now prove that $h^{-1} : S^{d-1} \mapsto \partial\tilde{K}$ is also sequential continuous. For every $\hat{x} \in S^{d-1}$, the point $h^{-1}(\hat{x})$ is the unique point of $\partial\tilde{K}$ which is hit by the ray defined by $\hat{x}$. Assume that h^{-1} is not continuous at some $\hat{a} \in S^{d-1}$. It means that we can find some $\epsilon_0 > 0$ and a sequence $\{\hat{x}_n\}_{n\geq 1} \subset S^{d-1}$, such that $\hat{x}_n \to \hat{a}$ and $||h^{-1}(\hat{x}_n) - h^{-1}(\hat{a})|| \geq \epsilon_0$. The vector $h^{-1}(\hat{x}_n)$ is parallel with $\hat{x}_n$ and the same is true for the pair $h^{-1}(\hat{a})$ and $\hat{a}$. Thus if n is large enough, the last inequality implies that either $||h^{-1}(\hat{x}_n)|| \leq ||h^{-1}(\hat{a})|| - \epsilon_0/2$ or $||h^{-1}(\hat{a})|| + \epsilon_0/2 \leq ||h^{-1}(\hat{x}_n)||$. Assume that there are infinitely many cases where the first situation holds true. Then if n is large enough, the point $h^{-1}(\hat{x}_n)$ enters in the cone $C(h^{-1}(\hat{a}))$ and must be an interior point of $\tilde{K}$, contradiction. In the other situation, $h^{-1}(\hat{a})$ would eventually become an interior element of the cone $C(h^{-1}(\hat{x}_n))$ for large enough n, again contradiction.

Let us define the map $\phi : \tilde{K} \mapsto \overline{B_1(0)}$ by $\phi(\mathbf{x}) := \mathbf{x}/||h^{-1}(\mathbf{x}/||\mathbf{x}||)||$ if $\mathbf{x} \neq 0$ and $\phi(0) = 0$. It is nothing but taking $\mathbf{x}$ and dividing it with the length of the segment between 0 and the point on the boundary corresponding to the ray generated by $\mathbf{x}/||\mathbf{x}||$. Clearly, ϕ is continuous. It is easy to check that the inverse of ϕ is given by $\phi^{-1} : \overline{B_1(0)} \mapsto \tilde{K}$ where $\phi^{-1}(\mathbf{y}) := \mathbf{y}||h^{-1}(\mathbf{y}/||\mathbf{y}||)||$ if $\mathbf{y} \neq 0$ and $\phi^{-1}(0) = 0$. This inverse is continuous because h^{-1} is continuous.

In conclusion, $\varphi := \phi \circ g : K \mapsto \overline{B_1(0)}$ is a homeomorphism, and we are done. □

Thus from now on we will assume without loss of generality that $K = \overline{B_1(0)}$.

Lemma 9.3. *Assume that* $\mathbf{f} : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ *is continuous with no fixed points. Then there exists a smooth function* $\tilde{f} : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ *with the same property.*

Proof. Our assumption says that $||\mathbf{f}(\mathbf{x}) - \mathbf{x}|| > 0$ for all $\mathbf{x} \in \overline{B_1(0)}$. The real valued map

$$\overline{B_1(0)} \ni \mathbf{x} \mapsto ||\mathbf{f}(\mathbf{x}) - \mathbf{x}|| \in \mathbb{R}$$

is continuous and defined on a compact set. Thus it attains its minimum in some point $\mathbf{x}_m$. It follows that:

$$||\mathbf{f}(\mathbf{x}) - \mathbf{x}|| \geq ||\mathbf{f}(\mathbf{x}_m) - \mathbf{x}_m|| = \epsilon_0 > 0. \tag{9.1}$$

Let us extend $\mathbf{f}$ to the whole of $\mathbb{R}^d$ in the following way. Define $\mathbf{g} : \mathbb{R}^d \mapsto \overline{B_1(0)}$ by $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ if $||\mathbf{x}|| \leq 1$, and $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}/||\mathbf{x}||)$ if $||\mathbf{x}|| > 1$. The extension is continuous, and $||\mathbf{g}(\mathbf{x})|| \leq 1$ for all $\mathbf{x}$.

Define the function $j : \mathbb{R}^d \mapsto \mathbb{R}$, $j(\mathbf{x}) = e^{-1/(1-||\mathbf{x}||^2)}$ if $||\mathbf{x}|| < 1$ and $j(\mathbf{x}) = 0$ if $||\mathbf{x}|| \geq 1$. The function j is non-negative, belongs to $C^\infty(\mathbb{R}^d)$ and has a positive integral $I := \int_{\mathbb{R}^d} j(\mathbf{x})d\mathbf{x} > 0$. Define $\tilde{j}(\mathbf{x}) := j(\mathbf{x})/I$. Then $\int_{\mathbb{R}^d} \tilde{j}(\mathbf{x})d\mathbf{x} = 1$.

Now if $\delta > 0$ we define the function $J_\delta(\mathbf{x}) := \delta^{-d}\,\tilde{j}(\delta^{-1}\mathbf{x})$. Clearly, J_δ is non-negative, belongs to $C^\infty(\mathbb{R}^d)$, it is non-zero only if $||\mathbf{x}|| < \delta$, and $\int_{\mathbb{R}^d} J_\delta(\mathbf{x})d\mathbf{x} = 1$ independently of δ.

Define the function $\mathbf{g}_\delta : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ by the formula:

$$\mathbf{g}_\delta(\mathbf{x}) := \int_{\mathbb{R}^d} J_\delta(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^d} \mathbf{g}(\mathbf{x}-\mathbf{y})J_\delta(\mathbf{y})d\mathbf{y}. \tag{9.2}$$

That the range of $\mathbf{g}_\delta$ is included in $\overline{B_1(0)}$ is a consequence of the fact that $||\mathbf{g}(\mathbf{y})|| \leq 1$ and $\int_{\mathbb{R}^d} J_\delta(\mathbf{x}-\mathbf{y})d\mathbf{y} = 1$ independently of $\mathbf{x}$. The function $\mathbf{g}_\delta$ is smooth because J_δ is smooth.

Now we can write:

$$\mathbf{g}_\delta(\mathbf{x}) - \mathbf{g}(\mathbf{x}) = \int_{\mathbb{R}^d} [\mathbf{g}(\mathbf{x}-\mathbf{y}) - \mathbf{g}(\mathbf{x})]J_\delta(\mathbf{y})d\mathbf{y} = \int_{||\mathbf{y}||\leq\delta} [\mathbf{g}(\mathbf{x}-\mathbf{y}) - \mathbf{g}(\mathbf{x})]J_\delta(\mathbf{y})d\mathbf{y}, \tag{9.3}$$

where the second equality comes from the support properties of J_δ. If we impose the condition $\delta < 1$, then $\mathbf{x}-\mathbf{y} \in \overline{B_2(0)}$ if $||\mathbf{y}|| \leq \delta$ and $||\mathbf{x}|| \leq 1$. The function $\mathbf{g}$ restricted to the compact set $\overline{B_2(0)}$ is uniformly continuous, thus there exists some $\delta_0 > 0$ small enough such that

$$||\mathbf{g}(\mathbf{x}') - \mathbf{g}(\mathbf{x}'')|| \leq \epsilon_0/2 \quad \text{whenever} \quad ||\mathbf{x}'-\mathbf{x}''|| \leq \delta_0, \quad \mathbf{x}', \mathbf{x}'' \in \overline{B_2(0)}.$$

Applying this estimate in (9.3) we obtain that $||\mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{g}(\mathbf{x})|| \leq \epsilon_0/2$, for all $\mathbf{x} \in \overline{B_1(0)}$. Using this in (9.1) it follows:

$$||\mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{x}|| \geq \epsilon_0/2 > 0, \quad \forall \mathbf{x} \in \overline{B_1(0)}. \tag{9.4}$$

The function $\mathbf{g}_{\delta_0}$ is our $\tilde{f}$ and the proof of this lemma is over. □

From now on we can assume that our function $\mathbf{f}$ is smooth and with no fixed points in $\overline{B_1(0)}$. The next lemma shows that such a function $\mathbf{f}$ would allow us to construct a smooth retraction of the unit ball onto its boundary.

Lemma 9.4. *Assume that* $\mathbf{f} : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ *is smooth with no fixed points. Then there exists a smooth function* $\mathbf{h} : \overline{B_1(0)} \mapsto S^{d-1}$ *such that* $h(\mathbf{x}) = \mathbf{x}$ *if* $\mathbf{x} \in S^{d-1}$.

Proof. We know that there exists some $\epsilon_0 > 0$ such that $||\mathbf{f}(\mathbf{x}) - \mathbf{x}|| \geq \epsilon_0$ for all $\mathbf{x} \in \overline{B_1(0)}$. We define the unit vector $\mathbf{w}(\mathbf{x}) := (||\mathbf{x} - \mathbf{f}(\mathbf{x})||)^{-1}\,(\mathbf{x} - \mathbf{f}(\mathbf{x}))$ which defines the direction of a straight line starting in $\mathbf{f}(\mathbf{x})$ and going through $\mathbf{x}$. This line is parametrized as $\mathbf{f}(\mathbf{x}) + t\mathbf{w}(\mathbf{x})$, with $t \geq 0$. The value $t = ||\mathbf{x} - \mathbf{f}(\mathbf{x})||$ gives $\mathbf{x}$. For even larger values of t we approach the boundary. There exists a unique positive value of $t(\mathbf{x}) \geq ||\mathbf{x} - \mathbf{f}(\mathbf{x})|| \geq \epsilon_0$ which corresponds to the intersection of this line with the unit sphere S^{d-1}. Namely, from the condition $||\mathbf{f}(\mathbf{x}) + t\mathbf{w}(\mathbf{x})||^2 = 1$ we obtain:

$$t(\mathbf{x}) = -\mathbf{f}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) + \sqrt{(\mathbf{f}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}))^2 + 1 - ||\mathbf{f}(\mathbf{x})||^2} \geq ||\mathbf{x} - \mathbf{f}(\mathbf{x})||,$$

where $\mathbf{f}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})$ is the inner product in $\mathbb{R}^d$. The only problem related to the smoothness of this function could appear if the square root can be zero. The square root is zero if $||\mathbf{f}(\mathbf{x})|| = 1$ and $0 = \mathbf{f}(\mathbf{x}) \cdot \mathbf{w}(x)$. Equivalently, $\mathbf{f}(\mathbf{x}) \cdot \mathbf{x} = 1$. The last equality demands that $\mathbf{x} = \mathbf{f}(\mathbf{x})$, both vectors having unit lenght and sitting on the boundary, situation excluded by our assumption of absence of fixed points. Thus $t(\mathbf{x})$ is smooth, and we can define

$$\mathbf{h}(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + t(\mathbf{x})\;\mathbf{w}(\mathbf{x}) \in S^{d-1}$$

which ends the proof.

□

Lemma 9.5. *Assume that* $\mathbf{h} : \overline{B_1(0)} \mapsto S^{d-1}$ *is smooth and* $\mathbf{h}(\mathbf{x}) = \mathbf{x}$ *if* $\mathbf{x} \in S^{d-1}$. *If* $0 \leq s \leq 1$, *define the map* $\mathbf{g}_s : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ *given by* $\mathbf{g}_s(\mathbf{x}) = (1-s)\mathbf{x} + s\mathbf{h}(\mathbf{x})$. *Then there exists* $0 < s_0 < 1$ *such that* $\mathbf{g}_s$ *is a bijection for all* $0 \leq s \leq s_0$.

Proof. First of all, we note that if $\mathbf{x} \in S^{d-1}$ then $\mathbf{g}_s(\mathbf{x}) = \mathbf{x}$. Thus the only thing we need to show is that $\mathbf{g}_s$ is injective and $\mathbf{g}_s(B_1(0)) = B_1(0)$.

For the injectivity part: consider the equality $\mathbf{g}_s(\mathbf{x}) = \mathbf{g}_s(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \overline{B_1(0)}$. This can be rewritten as:

$$\mathbf{x} - \mathbf{y} = -\frac{s}{1-s}(\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y})).$$

Reasoning as in Lemma 7.3 we can find a constant $C_h > 0$ such that $||\mathbf{h}(\mathbf{u}) - \mathbf{h}(\mathbf{w})|| \leq C_h ||\mathbf{u} - \mathbf{w}||$ for all $\mathbf{u}, \mathbf{w} \in \overline{B_1(0)}$. Thus we obtain:

$$||\mathbf{x} - \mathbf{y}|| \leq \frac{C_h s}{1-s} ||\mathbf{x} - \mathbf{y}||$$

which imposes $\mathbf{x} = \mathbf{y}$ if s is smaller than some small enough value $0 < \tilde{s} < 1$.

Now let us assume that $0 \leq s \leq \tilde{s}$. We want to prove that there exists $0 < s_0 \leq \tilde{s}$ such that $\mathbf{g}_s(B_1(0)) = B_1(0)$ for all $0 \leq s \leq s_0$.

One inclusion is easy: if $||\mathbf{x}|| < 1$, then $||\mathbf{g}_s(\mathbf{x})|| \leq (1-s)||\mathbf{x}|| + s < 1$. Thus $\mathbf{g}_s(B_1(0)) \subset B_1(0)$.

The other inclusion is more complicated. Let us consider the equation $\mathbf{g}_s(\mathbf{x}) = \mathbf{z}$, where $||\mathbf{z}|| \leq 1/4$ is arbitrary. This equation can be rewritten as $\mathbf{x} = (1-s)^{-1}\{\mathbf{z} - s\mathbf{h}(\mathbf{x})\}$. Now if s is smaller than some small enough value s_1, the vector $T(\mathbf{x}) := (1-s)^{-1}\mathbf{z} - s(1-s)^{-1}\mathbf{h}(\mathbf{x})$ obeys $||T(\mathbf{x})|| \leq 1/2$ for all $||\mathbf{x}|| \leq 1$. In particular, T invariates $\overline{B_{\frac{1}{2}}(0)}$. Also:

$$||T(\mathbf{u}) - T(\mathbf{w})|| \leq C_h \; s \; ||\mathbf{u} - \mathbf{w}||, \quad \forall \mathbf{u}, \mathbf{w} \in \overline{B_{\frac{1}{2}}(0)}.$$

Thus if $s < s_2 := \min\{s_1, C_h^{-1}\}$, the map T is a contraction and has a unique fixed point. This fixed point solves the equation $\mathbf{g}_s(\mathbf{x}) = \mathbf{z}$. Thus until now we showed that

$$\overline{B_{\frac{1}{4}}(0)} \subset \mathbf{g}_s(B_1(0)), \quad 0 \leq s < s_2 < 1.$$

Another important observation which we have to prove is that $\mathbf{g}_s(B_1(0))$ is an open set. Indeed, we have $[D\mathbf{g}_s(\mathbf{x})] = (1-s)I_{d\times d} + s[Dh(\mathbf{x})]$ and $\det[D\mathbf{g}_s(\mathbf{x})] \geq 1/2$ if s is smaller than some small enough s_3, for all $\mathbf{x} \in B_1(0)$; let $\mathbf{y} = \mathbf{g}_s(\mathbf{a})$ for some $\mathbf{a} \in B_1(0)$. Then from Theorem 8.3 (i) it follows that there exists some r small enough such that $\mathbf{g}_s(B_r(\mathbf{a}))$ is open, and since $\mathbf{y} \in \mathbf{g}_s(B_r(\mathbf{a}))$ there exists $\epsilon > 0$ so that $B_\epsilon(\mathbf{g}_s(\mathbf{a})) \subset \mathbf{g}_s(B_r(\mathbf{a})) \subset \mathbf{g}_s(B_1(0))$.

Now fix $0 < s_0 < \min\{s_2, s_3\}$. For $0 \leq s \leq s_0$ we know that $\mathbf{g}_s(B_1(0))$ is open and $\overline{B_{\frac{1}{4}}(0)} \subset \mathbf{g}_s(B_1(0)) \subset B_1(0)$. We need to show that $B_1(0) \subset \mathbf{g}_s(B_1(0))$.

Assume the contrary: there exists some $\mathbf{y}_0 \in B_1(0)$ which does not belong to $\mathbf{g}_s(B_1(0))$. Denote by I the closed segment joining 0 with $\mathbf{y}_0$. The set $E := I \cap \mathbf{g}_s(B_1(0))$ is not empty. Moreover, the set:

$$\{||\mathbf{y}|| : \; \mathbf{y} \in I \cap \mathbf{g}_s(B_1(0))\} \subset [0, ||\mathbf{y}_0||]$$

is not empty, and has a supremum $c < 1$. The supremum is always a limit point, hence there exists a sequence $\{\mathbf{y}_n\}_{n\geq 1} \subset I \cap \mathbf{g}_s(B_1(0))$ such that $||y_n|| \to c$. Because I is compact, there exists a subsequence $\mathbf{y}_{n_k}$ which converges in I to some point $\tilde{\mathbf{y}} \in I$, thus $\tilde{\mathbf{y}}$ is an adherent point of $\mathbf{g}_s(B_1(0))$ and $||\tilde{\mathbf{y}}|| = c < 1$. Clearly, $\tilde{\mathbf{y}} \notin \mathbf{g}_s(B_1(0))$ because otherwise, since $\mathbf{g}_s(B_1(0))$ is open, we could find elements of $I \cap \mathbf{g}_s(B_1(0))$ even farther away from the origin, contradicting the maximality of the length of $\tilde{\mathbf{y}}$.

Thus we have constructed $\tilde{\mathbf{y}} \in \overline{\mathbf{g}_s(B_1(0))} \setminus \mathbf{g}_s(B_1(0))$ with $||\tilde{\mathbf{y}}|| \leq ||\mathbf{y}_0|| < 1$. Being an adherent point of $\mathbf{g}_s(B_1(0))$, there must exist a sequence $\{\mathbf{z}_n\}_{n\geq 1} \subset \mathbf{g}_s(B_1(0))$ such that $\mathbf{z}_n \to \tilde{\mathbf{y}}$. Thus there exists a sequence $\{\mathbf{x}_n\}_{n\geq 1} \subset B_1(0)$ such that $\mathbf{g}_s(\mathbf{x}_n) = \mathbf{z}_n$. We can find a subsequence $\mathbf{x}_{n_k}$ which converges to some $\mathbf{x}_0 \in \overline{B_1(0)}$. Since $\mathbf{g}_s(\mathbf{x}_{n_k}) = \mathbf{z}_{n_k} \to \tilde{\mathbf{y}}$ and due to the continuity of $\mathbf{g}_s$, we must have $\mathbf{g}_s(\mathbf{x}_0) = \tilde{\mathbf{y}}$. But since $\tilde{\mathbf{y}} \notin \mathbf{g}_s(B_1(0))$, it must be that $\mathbf{x}_0 \in S^{d-1}$. But on the boundary,

$\mathbf{g}_s(\mathbf{x}_0) = \mathbf{x}_0$ and has unit length, which contradicts our assumption that $||\tilde{\mathbf{y}}|| \leq ||\mathbf{y}_0|| < 1$. Therefore, $\mathbf{y}_0$ cannot exist, and $B_1(0) \subset g_s(B_1(0))$.

□

We are finally ready to prove Brouwer's theorem. In the previous lemma we considered the smooth map $\mathbf{g}_s : \overline{B_1(0)} \mapsto \overline{B_1(0)}$. Define the function:

$$F(s) := \int_{B_1(0)} \det[D\mathbf{g}_s(\mathbf{x})]\, d\mathbf{x}, \quad 0 \leq s \leq 1.$$

The determinant of the Jacobi matrix $[D\mathbf{g}_s(\mathbf{x})]$ is a polynomial in s, thus $F(s)$ is a polynomial. Moreover, we have shown that if $0 \leq s \leq s_0$, the map $\mathbf{g}_s$ is nothing but a smooth and bijective change of coordinates in $B_1(0)$ with $\det[D\mathbf{g}_s(\mathbf{x})] > 0$, thus $F(s)$ is constant on $[0, s_0]$ and equal to the volume of $B_1(0)$. But if a polynomial is locally constant, then it is constant everywhere. Thus $F(1)$ should also be equal to the volume of $B_1(0)$.

Now let us show that this cannot be true. If $s = 1$, then $\mathbf{g}_1(\mathbf{x}) = \mathbf{h}(\mathbf{x})$ on $B_1(0)$. It means that

$$1 = ||\mathbf{h}(\mathbf{x})||^2 = \mathbf{g}_1(\mathbf{x}) \cdot \mathbf{g}_1(\mathbf{x}) = \sum_{k=1}^{d} (\mathbf{g}_1(\mathbf{x}))_k^2.$$

Differentiating with respect to x_j we obtain

$$0 = \sum_{k=1}^{d} [\partial_j (\mathbf{g}_1(\mathbf{x}))_k]\ (\mathbf{g}_1(\mathbf{x}))_k, \quad 1 \leq j \leq d,$$

or $[D\mathbf{g}_1(\mathbf{x})]^* \mathbf{g}_1(\mathbf{x}) = 0$ for all $\mathbf{x}$. Since $||\mathbf{g}(\mathbf{x})|| = 1$, we have that $[D\mathbf{g}_1(\mathbf{x})]^*$ is not injective, thus not invertible, hence with zero determinant. Therefore $\det[D\mathbf{g}_1(\mathbf{x})] = \det[D\mathbf{g}_1(\mathbf{x})]^* = 0$ for all $\mathbf{x}$, and $F(1) = 0 \neq \mathrm{vol}(B_1(0))$. This contradiction can be traced back to our assumption which claimed that $\mathbf{f}$ had no fixed points. The proof is over. □

10 Schauder's fixed point theorem

Theorem 10.1. *Let X be a Banach space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f \colon K \mapsto K$ there exists $x \in K$ such that $f(x) = x$.*

Proof. Given $\epsilon > 0$, the family of open sets $\{B_\epsilon(x) \colon x \in K\}$ is an open covering of K. Because K is compact, there exists a finite subcover, i.e. there exists N points $p_1, \ldots, p_N$ of K such that the balls $B_\epsilon(p_i)$ cover the whole set K.

Let K_ϵ be the convex hull of $p_1, \ldots, p_N$, defined by:

$$K_\epsilon := \left\{ \sum_{j=1}^{N} t_j p_j, \quad \sum_{j=1}^{N} t_j = 1,\ t_j \geq 0 \right\} \subset K.$$

It is an easy computation to show that K_ϵ is a convex set. Moreover, K_ϵ is immersed in an at most $N-1$ dimensional Euclidean space generated by the vectors $p_j - p_1$, where $j \in \{2, 3, \ldots, N\}$.

Define the function $g_j : K \mapsto \mathbb{R}_+$ by $g_j(x) = \epsilon - ||x - p_j||$ if $x \in B_\epsilon(p_j)$, and $g_j(x) = 0$ otherwise. Each function g_j is continuous, while $g(x) = \sum_{j=1}^{N} g_j(x)$ is positive due to the fact that any x has to be in some ball, where the corresponding g_j is positive. Since g is continuous and K compact, there exists $\delta > 0$ such that $g(x) \geq \delta$ for every $x \in K$.

Now consider the continuous map $\pi_\epsilon \colon K \to K_\epsilon$ given by:

$$\pi_\epsilon(x) := \sum_{j=1}^{N} \frac{g_j(x)}{g(x)}\, p_j, \quad \sum_{j=1}^{N} \frac{g_j(x)}{g(x)} = 1.$$

Since $||g_j(x)(x-p_j)|| \leq g_j(x)\epsilon$ for all j, we have:

$$\|\pi_\epsilon(x) - x\| \leq \sum_{j=1}^{N} \frac{||g_j(x)(p_j - x)||}{g(x)} \leq \epsilon, \quad \forall x \in K. \tag{10.1}$$

Now we define:

$$f_\epsilon : K_\epsilon \to K_\epsilon, \quad f_\epsilon(x) = \pi_\epsilon(f(x)).$$

This is a continuous function defined on a convex and compact set K_ϵ in a finite dimensional vector space. By Brouwer's fixed point theorem it admits a fixed point x_ϵ

$$f_\epsilon(x_\epsilon) = x_\epsilon.$$

Using (10.1) we get:

$$||\pi_\epsilon(f(x_\epsilon)) - f(x_\epsilon)|| \leq \epsilon,$$

thus for every $\epsilon > 0$ we have constructed $x_\epsilon \in K_\epsilon \subset K$ such that $||f(x_\epsilon) - x_\epsilon|| \leq \epsilon$.

Choosing $1/n$ instead of ϵ, we construct a sequence $\{x_n\}_{n\geq 1} \subset K$ such that $||f(x_n)-x_n|| \leq 1/n$. Since K is sequentially compact, we can find a subsequence x_{n_k} which converges to some point $\bar{x} \in K$ when $k \to \infty$. By writing:

$$||f(\bar{x}) - \bar{x}|| \leq ||f(\bar{x}) - f(x_{n_k})|| + ||f(x_{n_k}) - x_{n_k}|| + ||x_{n_k} - \bar{x}||, \quad k \geq 1,$$

we observe that due to the continuity of f at $\bar{x}$, the right hand side tends to zero with k. Thus $f(\bar{x}) = \bar{x}$ and we are done. □

11 Kakutani's fixed point theorem

Let $A \subset \mathbb{R}^d$ be a closed set, and denote by 2^A the set of all subsets of A. We say that $F : A \mapsto 2^A$ is upper semi-continuous if the following property holds: assume that $\{\mathbf{x}_n\}_{n\geq 1} \subset A$ with $\mathbf{x}_n \to \mathbf{x}_\infty \in A$, $\{\mathbf{y}_n\}_{n\geq 1} \subset A$ with $\mathbf{y}_n \to \mathbf{y}_\infty \in A$, and $\mathbf{y}_n \in F(\mathbf{x}_n)$; then we must have $\mathbf{y}_\infty \in F(\mathbf{x}_\infty)$.

Note that if we choose $\mathbf{x}_n = \mathbf{x}_\infty$ for all n, then the upper semi-continuity implies that if $\mathbf{y}_n \to \mathbf{y}_\infty \in A$ and $\mathbf{y}_n \in F(\mathbf{x}_\infty)$, then $\mathbf{y}_\infty \in F(\mathbf{x}_\infty)$. In other words, $F(\mathbf{x}_\infty)$ is always closed for all $\mathbf{x}_\infty \in A$.

Theorem 11.1. *Let $K \subset \mathbb{R}^d$ be a convex body. Let $F : K \mapsto 2^K$ be upper semicontinuous, such that $F(\mathbf{x}) \subset K$ is convex and nonempty. Then there exists $\mathbf{x}^* \in K$ such that $\mathbf{x}^* \in F(\mathbf{x}^*)$.*

Proof. Since K is compact, for every $m \geq 1$ there exist N_m points denoted by $\{\mathbf{w}_{j,m}\}_{j=1}^{N_m}$ such that $K \subset \cup_{j=1}^{N_m} B_{\frac{1}{m}}(\mathbf{w}_{j,m})$. It is important for what follows to note that we may choose the points $\mathbf{w}_{j,m}$ such that each ball $B_{\frac{1}{m}}(\mathbf{x})$ contains at most $\mathcal{N}$ (only depending on the dimension d) points $\mathbf{w}_{j,m}$, independent of m and $\mathbf{x}$.

For every $1 \leq j \leq N_m$ we define a map $g_{j,m} : K \mapsto \mathbb{R}_+$ by $g_{j,m}(\mathbf{x}) = \frac{1}{m} - ||\mathbf{x} - \mathbf{w}_{j,m}||$ if $\mathbf{x} \in B_{\frac{1}{m}}(\mathbf{w}_{j,m})$, and $g_{j,m}(\mathbf{x}) = 0$ otherwise. Each function $g_{j,m}$ is continuous, while $g_m(\mathbf{x}) = \sum_{j=1}^{N_m} g_{j,m}(\mathbf{x})$ is positive due to the fact that any $\mathbf{x}$ has to be in some ball j, where the corresponding $g_{j,m}$ is positive. Since g_m is continuous and K compact, there exists $\delta_m > 0$ such that $g_m(\mathbf{x}) \geq \delta_m$ for every $\mathbf{x} \in K$.

For every $1 \leq j \leq N_m$ we choose some $\mathbf{y}_{j,m} \in F(\mathbf{w}_{j,m}) \subset K$ in an arbitrary way. Define the map

$$\mathbf{f}_m : K \mapsto K, \quad \mathbf{f}_m(\mathbf{x}) := \sum_{j=1}^{N_m} \frac{g_{j,m}(\mathbf{x})}{g_m(\mathbf{x})}\, \mathbf{y}_{j,m}, \quad \sum_{j=1}^{N_m} \frac{g_{j,m}(\mathbf{x})}{g_m(\mathbf{x})} = 1, \quad \mathbf{y}_j \in F(\mathbf{w}_{j,m}).$$

The function $\mathbf{f}_m$ is continuous and defined on a convex body, thus Brouwer's fixed point theorem provides us with a fixed point $\mathbf{x}_m \in K$ such that $\mathbf{f}_m(\mathbf{x}_m) = \sum_{j=1}^{N_m} \frac{g_{j,m}(\mathbf{x}_m)}{g_m(\mathbf{x}_m)} \mathbf{y}_{j,m} = \mathbf{x}_m$, for every $m \geq 1$.

Because K is sequentially compact, we may find a subsequence $\mathbf{x}_{m_k}$ which converges to some $\mathbf{x}^* \in K$ when $k \to \infty$. It is important to note that when j varies from 1 to N_{m_k} we have $g_{j,m_k}(\mathbf{x}_{m_k}) \neq 0$ only for those $\mathbf{w}_{j,m_k}$ which obey $||\mathbf{x}_{m_k} - \mathbf{w}_{j,m_k}|| \leq \frac{1}{m_k} \leq \frac{1}{k}$. There are at most $\mathcal{N}$ indexes j such that $g_{j,m_k}(\mathbf{x}_{m_k}) \neq 0$:

$$\mathbf{f}_{m_k}(\mathbf{x}_{m_k}) = \mathbf{x}_{m_k} = \sum_{||\mathbf{x}_{m_k} - \mathbf{w}_{j,m_k}|| \leq \frac{1}{m_k}} \frac{g_{j,m_k}(\mathbf{x}_{m_k})}{g_{m_k}(\mathbf{x}_{m_k})} \mathbf{y}_{j,m_k}.$$

Thus for a fixed $k \geq 1$ we have finitely many $\mathbf{w}_{j,m_k}$ (at most $\mathcal{N}$, independently of k) which all lie in a small ball of radius $1/m_k$ around $\mathbf{x}_{m_k}$. We can reorganize the $\mathcal{N}$ closest points $\mathbf{w}_{j,m_k}$ and their corresponding $\mathbf{y}_{j,m_k}$ as $\mathcal{N}$ pairs of sequences $\{\tilde{\mathbf{w}}_k^s\}_{k \geq 1}$ and $\tilde{\mathbf{y}}_k^s$, with $\lim_{k\to\infty} ||\tilde{\mathbf{w}}_k^s - \mathbf{x}^*|| = 0$ and $\tilde{\mathbf{y}}_k^s \in F(\tilde{\mathbf{w}}_k^s)$ for all $1 \leq s \leq \mathcal{N}$. With the new notation:

$$\mathbf{f}_{m_k}(\mathbf{x}_{m_k}) = \mathbf{x}_{m_k} = \sum_{s=1}^{\mathcal{N}} \frac{g_{s,m_k}(\mathbf{x}_{m_k})}{g_{m_k}(\mathbf{x}_{m_k})} \tilde{\mathbf{y}}_k^s, \quad \lim_{k\to\infty} \tilde{\mathbf{w}}_k^s = \mathbf{x}^*.$$

Now we can choose a subsequence $\tilde{\mathbf{y}}_{k_n}^s$, $n \geq 1$, such that $\tilde{\mathbf{y}}_{k_n}^s$ converges to some $\mathbf{y}^s \in K$. Thus:

$$\mathbf{x}_{m_{k_n}} = \sum_{s=1}^{\mathcal{N}} \frac{g_{s,m_{k_n}}(\mathbf{x}_{m_{k_n}})}{g_{m_{k_n}}(\mathbf{x}_{m_{k_n}})} \tilde{\mathbf{y}}_{k_n}^s, \quad \lim_{n\to\infty} \tilde{\mathbf{w}}_{k_n}^s = \mathbf{x}^*, \quad \lim_{n\to\infty} \tilde{\mathbf{y}}_{k_n}^s = \mathbf{y}^s.$$

To summarize, we have $\lim_{n\to\infty} \tilde{\mathbf{w}}_{k_n}^s = \mathbf{x}^*$, $\lim_{n\to\infty} \tilde{\mathbf{y}}_{k_n}^s = \mathbf{y}^s$, and $\tilde{\mathbf{y}}_{k_n}^s \in F(\tilde{\mathbf{w}}_{k_n}^s)$. Because F is upper semi-continuous, we must have $\mathbf{y}^s \in F(\mathbf{x}^*)$ for all $1 \leq s \leq \mathcal{N}$. Moreover,

$$\left\| \mathbf{x}_{m_{k_n}} - \sum_{s=1}^{\mathcal{N}} \frac{g_{s,m_{k_n}}(\mathbf{x}_{m_{k_n}})}{g_{m_{k_n}}(\mathbf{x}_{m_{k_n}})} \mathbf{y}^s \right\| \leq \max_{s=1}^{\mathcal{N}} ||\tilde{\mathbf{y}}_{k_n}^s - \mathbf{y}^s|| \to 0.$$

Since $F(\mathbf{x}^*)$ is convex, and all $\mathbf{y}^s \in F(\mathbf{x}^*)$, the convex combination $\sum_{s=1}^{\mathcal{N}} \frac{g_{s,m_{k_n}}(\mathbf{x}_{m_{k_n}})}{g_{m_{k_n}}(\mathbf{x}_{m_{k_n}})} \mathbf{y}^s$ is an element of $F(\mathbf{x}^*)$. Since $\mathbf{x}_{m_{k_n}} \to \mathbf{x}^*$, we have that $\mathbf{x}^*$ must be an adherent point of $F(\mathbf{x}^*)$. Since $F(\mathbf{x}^*)$ is closed, then $\mathbf{x}^* \in F(\mathbf{x}^*)$ and we are done.

□

12 Existence of Nash equilibrium for finite games with two players

Let $K_1 \subset \mathbb{R}^{d_1}$ and $K_2 \subset \mathbb{R}^{d_2}$ be two convex bodies (i.e. convex, compact and with non-empty interior). The set $K = K_1 \times K_2 \subset \mathbb{R}^{d_1+d_2}$ is also a convex body. Let $\phi : K \mapsto \mathbb{R}$ be a continuous function. We say that ϕ is concave in the first variable if for every $\mathbf{x}, \mathbf{y} \in K_1$ and for every $0 \leq t \leq 1$ and $\mathbf{z} \in K_2$ we have:

$$\phi(t\mathbf{x} + (1-t)\mathbf{y}, \mathbf{z}) \geq t\phi(\mathbf{x}, \mathbf{z}) + (1-t)\phi(\mathbf{y}, \mathbf{z}).$$

We say that ϕ is strictly concave in the first variable if for every $\mathbf{x} \neq \mathbf{y}$ and $0 < t < 1$ we have:

$$\phi(t\mathbf{x} + (1-t)\mathbf{y}, \mathbf{z}) > t\phi(\mathbf{x}, \mathbf{z}) + (1-t)\phi(\mathbf{y}, \mathbf{z}).$$

Obvious definitions apply for the second variable.

Let ϕ_1 and ϕ_2 be two real valued continuous functions defined on K, such that ϕ_1 is concave in the first variable while ϕ_2 is concave in the second variable. They model the payoff functions of two players. The triple (ϕ_1, ϕ_2, K) is called a *finite game* with two players.

A point $[\mathbf{x}^*, \mathbf{y}^*] \in K$ is called a *Nash equilibrium* if for every $\mathbf{x} \in K_1$ and $\mathbf{y} \in K_2$ one has:

$$\phi_1(\mathbf{x}, \mathbf{y}^*) \leq \phi_1(\mathbf{x}^*, \mathbf{y}^*) \quad \text{and} \quad \phi_2(\mathbf{x}^*, \mathbf{y}) \leq \phi_2(\mathbf{x}^*, \mathbf{y}^*).$$

Here is the main result of this section:

Theorem 12.1. *Every finite game admits a Nash equilibrium.*

In order to prove the theorem, we will need two technical lemmas.

Lemma 12.2. *Let $\phi : K \mapsto \mathbb{R}$ be concave in the first variable. Fix $\mathbf{x}_0 \in K_1$ and $\mathbf{y}_0 \in K_2$. Then the function $F(\mathbf{x}) := \phi(\mathbf{x}, \mathbf{y}_0) - ||\mathbf{x} - \mathbf{x}_0||^2$ is strictly concave on K_1.*

Proof. It is enough to show that $g(\mathbf{x}) := -||\mathbf{x}-\mathbf{x}_0||^2$ is strictly concave. Let $t \in (0, 1)$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. Then it is easy to compute:

$$g(t\mathbf{x}_1 + (1-t)\mathbf{x}_2) - [tg(\mathbf{x}_1) + (1-t)g(\mathbf{x}_2)] = t(1-t)||\mathbf{x}_1 - \mathbf{x}_2||^2 > 0.$$

□

Lemma 12.3. *Let $\phi : K \mapsto \mathbb{R}$ be continuous and concave in the first variable. Fix $\mathbf{x}_0 \in K_1$ and $\mathbf{y}_0 \in K_2$. Then the function $\phi(\mathbf{x}, \mathbf{y}_0) - ||\mathbf{x}-\mathbf{x}_0||^2$ attains its maximum in a unique point $\mathbf{x}(\mathbf{x}_0, \mathbf{y}_0)$, and the map*

$$K \ni [\mathbf{x}_0, \mathbf{y}_0] \mapsto \mathbf{x}(\mathbf{x}_0, \mathbf{y}_0) \in K_1$$

is continuous. Finally, if $\mathbf{x}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{x}_0$ then

$$\phi(\mathbf{x}, \mathbf{y}_0) \leq \phi(\mathbf{x}_0, \mathbf{y}_0), \quad \forall \mathbf{x} \in K_1. \tag{12.1}$$

Proof. Since ϕ is jointly continuous on $K_1 \times K_2$, it is also separately continuous. For a given pair $[\mathbf{x}_0, \mathbf{y}_0] =: \mathbf{z} \in K$ we define the real valued map $f_\mathbf{z} : K_1 \mapsto \mathbb{R}$ given by $f_\mathbf{z}(\mathbf{x}) := \phi(\mathbf{x}, \mathbf{y}_0) - ||\mathbf{x}-\mathbf{x}_0||^2$. This function is continuous hence we know that it attains its maximum $M < \infty$ because K_1 is compact. Moreover, from the previous lemma we know that $f_\mathbf{z}$ is strictly concave.

Let us first show that the value M is attained in only one point. Assume the contrary: there exist two points $\mathbf{x}_1 \neq \mathbf{x}_2 \in K_1$ such that $f_\mathbf{z}(\mathbf{x}) \leq f_\mathbf{z}(\mathbf{x}_1) = f_\mathbf{z}(\mathbf{x}_2) = M$ for all $\mathbf{x} \in K_1$. But since $f_\mathbf{z}$ is strictly concave we have:

$$f_\mathbf{z}(\mathbf{x}_1/2 + \mathbf{x}_2/2) > \frac{1}{2} f_\mathbf{z}(\mathbf{x}_1) + \frac{1}{2} f_\mathbf{z}(\mathbf{x}_2) = M$$

which contradicts the fact that M is the maximal value.

Thus we may denote by $\mathbf{x}(\mathbf{z}) \in K_1$ the unique point which maximizes $f_\mathbf{z}$:

$$f_\mathbf{z}(\mathbf{x}) \leq f_\mathbf{z}(\mathbf{x}(\mathbf{z})), \quad \forall \mathbf{x} \in K_1.$$

We now show that $\mathbf{x}(\mathbf{z})$ is continuous on K. Assume the contrary: there exists some $\mathbf{u} \in K$ such that $\mathbf{x}(\cdot)$ is not sequentially continuous at $\mathbf{u}$. Then we may find some $\epsilon_0 > 0$ and a sequence $\{\mathbf{z}_n\}_{n\geq 1} \subset K$ such that $\mathbf{z}_n \to \mathbf{u}$ and $|\mathbf{x}(\mathbf{z}_n) - \mathbf{x}(\mathbf{u})| \geq \epsilon_0$. Since K_1 is sequentially compact, we can find a subsequence $\{\mathbf{x}(\mathbf{z}_{n_k})\}_{k\geq 1}$ which converges to some point $\mathbf{w} \in K_1$. Clearly, $\mathbf{w} \neq \mathbf{x}(\mathbf{u})$. We have the inequality:

$$f_{\mathbf{z}_{n_k}}(\mathbf{x}(\mathbf{z}_{n_k})) \geq f_{\mathbf{z}_{n_k}}(\mathbf{x}(\mathbf{u})), \quad \forall k \geq 1.$$

Now the map:

$$K \times K_1 \ni [\mathbf{z}, \mathbf{x}] \mapsto f_\mathbf{z}(\mathbf{x}) = \phi(\mathbf{x}, \mathbf{y}_0) - ||\mathbf{x} - \mathbf{x}_0||^2 \in \mathbb{R}$$

is jointly continuous being the sum of two jointly continuous functions. Using this continuity and taking k to infinity we obtain $f_\mathbf{u}(\mathbf{w}) \geq f_\mathbf{u}(\mathbf{x}(\mathbf{u}))$ which shows that we must have $\mathbf{w} = \mathbf{x}(\mathbf{u})$ because the maximum of $f_\mathbf{u}$ is taken at only one point, contradiction.

Finally, let us prove (12.1). The assumption is that $\mathbf{x}(\mathbf{z}) = \mathbf{x}_0$. Let $t \in (0, 1)$. If $\mathbf{x} \in K_1$ we have:

$$f_\mathbf{z}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)) = \phi(t\mathbf{x} + (1-t)\mathbf{x}_0, \mathbf{y}_0) - t^2||\mathbf{x}-\mathbf{x}_0||^2 \geq t\phi(\mathbf{x}, \mathbf{y}_0) + (1-t)\phi(\mathbf{x}_0, \mathbf{y}_0) - t^2||\mathbf{x}-\mathbf{x}_0||^2$$

and

$$f_{\mathbf{z}}(\mathbf{x}_0) = \phi(\mathbf{x}_0, \mathbf{y}_0) \geq f_{\mathbf{z}}(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)).$$

From these two inequalities we obtain:

$$\phi(\mathbf{x}, \mathbf{y}_0) \leq \phi(\mathbf{x}_0, \mathbf{y}_0) + t||\mathbf{x} - \mathbf{x}_0||^2, \quad \forall t \in (0, 1).$$

Taking t to zero finishes the proof. □

Proof of Theorem 12.1. Applying Lemma 12.2 to ϕ_1 we can construct a continuous map

$$K \ni [\mathbf{x}_0, \mathbf{y}_0] \mapsto \mathbf{x}(\mathbf{x}_0, \mathbf{y}_0) \in K_1$$

such that $\mathbf{x}(\mathbf{x}_0, \mathbf{y}_0)$ is the unique point which maximizes $\phi_1(\mathbf{x}, \mathbf{y}_0) - ||\mathbf{x} - \mathbf{x}_0||^2$. Moreover, if $\mathbf{x}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{x}_0$ then $\phi_1(\mathbf{x}, \mathbf{y}_0) \leq \phi_1(\mathbf{x}_0, \mathbf{y}_0)$. By a completely analogous argument, we can consider the function $\phi_2(\mathbf{x}_0, \mathbf{y}) - ||\mathbf{y} - \mathbf{y}_0||^2$ which will admit a unique maximizing point $y(\mathbf{x}_0, \mathbf{y}_0)$, a point which also continuously depends on its arguments. Moreover, if $\mathbf{y}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{y}_0$ then $\phi_2(\mathbf{x}_0, \mathbf{y}) \leq \phi_2(\mathbf{x}_0, \mathbf{y}_0)$.

Thus the problem is reduced to finding a fixed point of the continuous map:

$$K \ni [\mathbf{x}_0, \mathbf{y}_0] \mapsto [\mathbf{x}(\mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(\mathbf{x}_0, \mathbf{y}_0)] \in K,$$

whose existence is insured by the Brouwer Fixed Point Theorem. The proof is over. □

13 The Hairy Ball Theorem

The question we want to answer here is of geometric nature: given the unit sphere $S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : \; ||\mathbf{x}|| = 1\} \subset \mathbb{R}^d$ for $d \geq 2$, is it possible to find a continuous tangent vector field $\tilde{\mathbf{w}} : S^{d-1} \mapsto \mathbb{R}^d$ which vanishes nowhere? If it is possible, then $||\tilde{\mathbf{w}}(\mathbf{x})||$ has a positive lower bound because S^{d-1} is compact, and in this way we would be able to construct a continuous vector field $\mathbf{w} : S^{d-1} \mapsto S^{d-1}$ given by $\mathbf{w}(\mathbf{x}) := \frac{1}{||\tilde{\mathbf{w}}(\mathbf{x})||}\tilde{\mathbf{w}}(\mathbf{x})$, which satisfies $\mathbf{w}(\mathbf{x}) \cdot \mathbf{x} = 0$.

If $d = 2p$ is even, then such a vector field exists and let us construct an example. If $\mathbf{x} = [x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{2p-1}, x_{2p}]$ then we can define $\mathbf{u} \in \mathbb{R}^p$ by $\mathbf{u} := [x_1, x_2, \ldots, x_p]$ and $\tilde{\mathbf{u}} \in \mathbb{R}^p$ by $\tilde{\mathbf{u}} := [x_{p+1}, \ldots, x_{2p-1}, x_{2p}]$. Thus $\mathbf{x} = [\mathbf{u}, \tilde{\mathbf{u}}]$. Define $\mathbf{w}(\mathbf{x}) = [-\tilde{\mathbf{u}}, \mathbf{u}]$. Then clearly $||\mathbf{w}(\mathbf{x})|| = ||\mathbf{x}|| = 1$ and $\mathbf{w}(\mathbf{x}) \cdot \mathbf{x} = 0$.

Theorem 13.1. *If $d \geq 3$ is odd, then one cannot construct a continuous map $\mathbf{w} : S^{d-1} \mapsto S^{d-1}$ such that $\mathbf{w}(\mathbf{x}) \cdot \mathbf{x} = 0$ on S^{d-1}.*

Proof. We will assume that such a vector field exists, and then we will arrive at a contradiction.

Lemma 13.2. *Let $d \geq 2$. Assume that $\mathbf{w} : S^{d-1} \mapsto S^{d-1}$ is continuous and $\mathbf{w}(\mathbf{x}) \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in S^{d-1}$. Define $D_1 := \{\mathbf{x} \in \mathbb{R}^d : \frac{1}{2} < ||\mathbf{x}|| < \frac{3}{2}\}$. Then there exists a smooth map $\tilde{\mathbf{w}} : D_1 \mapsto \mathbb{R}^d$ such that $\tilde{\mathbf{w}}(D_1) \subset S^{d-1}$ and $\tilde{\mathbf{w}}(\mathbf{x}) \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in D_1$.*

Proof. If $\mathbf{x} \neq 0$ we denote by $\hat{\mathbf{x}} := \frac{1}{||\mathbf{x}||}\mathbf{x} \in S^{d-1}$. Define $D_2 := \{\mathbf{x} \in \mathbb{R}^d : \; \frac{1}{4} \leq ||\mathbf{x}|| \leq 2\}$, and consider the function $\mathbf{w}_2 : D_2 \mapsto \mathbb{R}^d$ given by $\mathbf{w}_2(\mathbf{x}) := \mathbf{w}(\hat{\mathbf{x}})$. Then $\mathbf{w}_2$ is continuous on the compact set D_2, thus uniformly continuous. Given any $\epsilon > 0$ we may find $\delta > 0$ such that $||\mathbf{w}_2(\mathbf{x}) - \mathbf{w}_2(\mathbf{x}')|| < \epsilon$ as soon as $\mathbf{x}, \mathbf{x}' \in D_2$ and $||\mathbf{x} - \mathbf{x}'|| < \delta$. We will reason as in Lemma 9.3: consider the function J_δ and define as in (9.2)

$$\mathbf{g}_\delta(\mathbf{x}) := \int_{\mathbb{R}^d} J_\delta(\mathbf{x} - \mathbf{y})\mathbf{w}_2(\mathbf{y})d\mathbf{y} = \int_{\mathbb{R}^d} \mathbf{w}_2(\mathbf{x} - \mathbf{y})J_\delta(\mathbf{y})d\mathbf{y}, \quad \mathbf{x} \in D_1.$$

As in that lemma, we may write

$$\mathbf{g}_\delta(\mathbf{x}) - \mathbf{w}_2(\mathbf{x}) = \int_{\mathbb{R}^d} [\mathbf{w}_2(\mathbf{x} - \mathbf{y}) - \mathbf{w}_2(\mathbf{x})]J_\delta(\mathbf{y})d\mathbf{y} = \int_{||\mathbf{y}|| \leq \delta} [\mathbf{w}_2(\mathbf{x} - \mathbf{y}) - \mathbf{w}_2(\mathbf{x})]J_\delta(\mathbf{y})d\mathbf{y}.$$

If $\delta < 1/4$ then $\mathbf{x}-\mathbf{y} \in D_2$, hence using the uniform continuity of $\mathbf{w}_2$ on D_2 we may find $\delta_0 < 1/4$ small enough such that $||\mathbf{w}_2(\mathbf{x}-\mathbf{y}) - \mathbf{w}_2(\mathbf{x})|| \leq 1/10$ for every $\mathbf{x} \in D_1$ and $||\mathbf{y}|| \leq \delta_0$. This leads to

$$||\mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})|| \leq 1/10 \quad \text{and} \quad ||\mathbf{g}_{\delta_0}(\mathbf{x})|| \geq 9/10, \quad \forall \mathbf{x} \in D_1.$$

The function $\mathbf{w}_3 : D_1 \mapsto \mathbb{R}^d$ given by (remember that $\mathbf{w}_2(\mathbf{x}) \cdot \mathbf{x} = 0$)

$$\mathbf{w}_3(\mathbf{x}) := \mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{x}\, \frac{\mathbf{g}_{\delta_0}(\mathbf{x}) \cdot \mathbf{x}}{||\mathbf{x}||^2} = \mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{x}\, \frac{[\mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})] \cdot \mathbf{x}}{||\mathbf{x}||^2}$$

is smooth on D_1 and obeys:

$$\mathbf{w}_3(\mathbf{x}) \cdot \mathbf{x} = 0, \quad ||\mathbf{w}_3(\mathbf{x})|| \geq ||\mathbf{g}_{\delta_0}(\mathbf{x})|| - ||\mathbf{g}_{\delta_0}(\mathbf{x}) - \mathbf{w}_2(\mathbf{x})|| \geq 8/10, \quad \forall \mathbf{x} \in D_1.$$

Finally, we can define the function $\tilde{\mathbf{w}}(\mathbf{x}) := \frac{1}{||\mathbf{w}_3(\mathbf{x})||}\mathbf{w}_3(\mathbf{x}) \in S^{d-1}$ which is smooth and orthogonal on $\mathbf{x}$. Moreover, we have the estimate (see Lemma 7.3 for the notation):

$$||\Delta \tilde{\mathbf{w}}||_{\infty, \overline{D}_1} < \infty. \tag{13.1}$$

□

Define $D_3 := \{\mathbf{x} \in \mathbb{R}^d :\ \frac{9}{10} < ||\mathbf{x}|| < \frac{10}{9}\} \subset D_1$. If $s \in \mathbb{R}$ we denote by

$$E_s := \left\{ \mathbf{x} \in \mathbb{R}^d :\ \sqrt{\frac{81}{100} + s^2} < ||\mathbf{x}|| < \sqrt{\frac{100}{81} + s^2} \right\}.$$

Lemma 13.3. *Let $\mathbf{h}_s : D_3 \mapsto E_s$ given by $\mathbf{h}_s(\mathbf{x}) := \mathbf{x} + s\tilde{\mathbf{w}}(\mathbf{x})$. If $|s| > 0$ is sufficiently small, then the map $\mathbf{h}_s$ is a bijection.*

Proof. Because $||\mathbf{h}_s(\mathbf{x})||^2 = ||\mathbf{x}||^2 + s^2$ it is easy to see that $\mathbf{h}_s(D_3) \subset E_s$ for all s. We need to show that $\mathbf{h}_s$ is injective and surjective if s is small enough.

Let us start by showing injectivity. Assume that h_s is not injective in a neighborhood of $s = 0$. Then there exists a sequence $\{s_n\}_{n \geq 1}$ which converges to 0 such that for every $n \neq 1$ there exist $\mathbf{x}_n \neq \mathbf{y}_n \in D_3$ such that $\mathbf{h}_{s_n}(\mathbf{x}_n) = \mathbf{h}_{s_n}(\mathbf{y}_n)$. This is equivalent with $\mathbf{x}_n - \mathbf{y}_n = s_n[\tilde{\mathbf{w}}(\mathbf{y}_n) - \tilde{\mathbf{w}}(\mathbf{x}_n)]$, which implies that $||\mathbf{x}_n - \mathbf{y}_n|| \leq 2|s_n|$. Since $D_3 \subset D_1$, if $|s_n|$ is small enough then the whole segment joining $\mathbf{x}_n$ and $\mathbf{y}_n$ is included in D_1. Using (13.1) and (7.3) for $\tilde{\mathbf{w}}$ we get that

$$||\mathbf{x}_n - \mathbf{y}_n|| = |s_n|\, ||\tilde{\mathbf{w}}(\mathbf{y}_n) - \tilde{\mathbf{w}}(\mathbf{x}_n)|| \leq |s_n|\, ||\Delta \tilde{\mathbf{w}}||_{\infty, \overline{D}_1} ||\mathbf{x}_n - \mathbf{y}_n||$$

which is incompatible with $\mathbf{x}_n \neq \mathbf{y}_n$ if $|s_n|$ is small enough.

Now we have to prove that $\mathbf{h}_s$ is surjective if $|s|$ is small enough. Let $\mathbf{y} \in E_s$. We have to show that the equation $\mathbf{h}_s(\mathbf{x}) = \mathbf{y}$ has a solution. This equation is equivalent with $\mathbf{x} = \mathbf{y} - s\tilde{\mathbf{w}}(\mathbf{x})$, which looks like a fixed point equation.

Define the closed set $D_4 := \{\mathbf{x} \in \mathbb{R}^d :\ \frac{8}{9} \leq ||\mathbf{x}|| \leq \frac{9}{8}\}$. If $|s|$ is sufficiently small, then $E_s \subset D_4$. Moreover, $D_3 \subset D_4 \subset D_1$.

The set D_4 is closed in $\mathbb{R}^d$, thus together with the induced Euclidean metric it forms a complete metric space. We want to show that the map $\mathbf{f}_{\mathbf{y},s} : D_4 \mapsto D_4$ given by $\mathbf{f}_{\mathbf{y},s}(\mathbf{x}) := \mathbf{y} - s\tilde{\mathbf{w}}(\mathbf{x})$ is a contraction on D_4 provided $|s|$ is small enough. If this is true, then the unique fixed point which obeys $\mathbf{f}_{\mathbf{y},s}(\mathbf{x}) = \mathbf{x}_y$ has the property that $||\mathbf{x}_y||^2 + s^2 = ||\mathbf{y}||^2$, thus $\mathbf{x}_y \in D_3$ if $\mathbf{y} \in E_s$.

Now let us show that $\mathbf{f}_{\mathbf{y},s}$ is a contraction for small $|s|$. First, using that $||\tilde{\mathbf{w}}(\mathbf{x})|| = 1$, then if $\frac{10}{9} + 2|s| \leq \frac{9}{8}$ and $\frac{8}{9} \leq \frac{9}{10} - |s|$ we have that $\mathbf{f}_{\mathbf{y},s}(D_4) \subset D_4$. Second, if $\mathbf{x}_1, \mathbf{x}_2 \in D_4$ with $||\mathbf{x}_1 - \mathbf{x}_2|| \geq \frac{1}{100}$, we have:

$$||\mathbf{f}_{\mathbf{y},s}(\mathbf{x}_1) - \mathbf{f}_{\mathbf{y},s}(\mathbf{x}_2)|| \leq 200\, |s|\, ||\mathbf{x}_1 - \mathbf{x}_2||, \quad \text{if} \quad ||\mathbf{x}_1 - \mathbf{x}_2|| \geq \frac{1}{100}.$$

Third, if $\mathbf{x}_1, \mathbf{x}_2 \in D_4$ with $||\mathbf{x}_1 - \mathbf{x}_2|| \leq \frac{1}{100}$, then since the ball $B_{\frac{1}{100}}(\mathbf{x}_1)$ is completely included in D_1, the straight segment joining $\mathbf{x}_1$ and $\mathbf{x}_2$ is included in D_1 and using again (7.3) we obtain:

$$||\mathbf{f}_{\mathbf{y},s}(\mathbf{x}_1) - \mathbf{f}_{\mathbf{y},s}(\mathbf{x}_2)|| \leq |s| \, ||\Delta\tilde{\mathbf{w}}||_{\infty,\overline{D}_1} \, ||\mathbf{x}_1 - \mathbf{x}_2||, \quad \text{if} \quad ||\mathbf{x}_1 - \mathbf{x}_2|| \leq \frac{1}{100}.$$

Thus $\mathbf{f}_{\mathbf{y},s}$ is a contraction if $|s|$ is smaller than some critical value s_0 which is independent of $\mathbf{y}$. This implies that the fixed point exists for all $\mathbf{y} \in E_s$ provided $0 \leq |s| \leq s_0$, and we are done. □

We are now ready to finish the proof of the Hairy Ball Theorem. Since the map $\mathbf{h}_s$ is a smooth bijection between D_3 and E_s and $\det[D\mathbf{h}_s](\mathbf{x}) > 0$ if $|s| \leq s_0$, we must have the equality:

$$\mathrm{Vol}(E_s) = \int_{D_3} \det[D\mathbf{h}_s](\mathbf{x}) d\mathbf{x}, \quad |s| \leq s_0.$$

The right hand side of the above equality is a polynomial in s of degree at most d. The left hand side can be calculated explicitly: it equals the difference of the volumes of two d-dimensional balls:

$$\mathrm{Vol}(E_s) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \left\{ \left(\frac{100}{81} + s^2 \right)^{\frac{d}{2}} - \left(\frac{81}{100} + s^2 \right)^{\frac{d}{2}} \right\}.$$

This function is analytic around $s = 0$, and since it equals a polynomial if $|s| \leq s_0$, by analytic continuation it must be a polynomial (thus analytic) for all $s \in \mathbb{C}$. The function $\left(\frac{100}{81} + s^2\right)^{\frac{d}{2}}$ is analytic if $|s| < 10/9$, thus $\left(\frac{81}{100} + s^2\right)^{\frac{d}{2}}$ must also be analytic on the same disk. Since $d = 2p+1$ is odd, the function $\left(\frac{81}{100} + s^2\right)^{\frac{d}{2}}$ can be factorized as the product $\left(\frac{81}{100} + s^2\right)^{p} \left(\frac{81}{100} + s^2\right)^{\frac{1}{2}}$. But this function is not analytic at $s = \frac{8i}{9}$, and this is our contradiction.

□

14 The Jordan Curve Theorem

14.1 Some preparatory results

Let (X, d) be a metric space with the topology generated by d. We say that X is not connected if we can find two non-empty open sets O_1 and O_2 which are disjoint and $X = O_1 \cup O_2$. A subset $A \subset X$ is not connected if the induced metric space (A, d) is not connected. If a metric space can be written as a union of connected sets $\{O_i\}$, then they are called the connected components of X.

Let $-\infty < a \leq b < \infty$ and let $\gamma : [a, b] \mapsto \mathbb{R}^2$ be a homeomorphism (continuous, invertible and with continuous inverse). Then $\gamma([a, b])$ is called an arc. A set $A \subset \mathbb{R}^2$ is called path-connected if given any two distinct points $\mathbf{x}, \mathbf{y} \in A$ we can find an arc joining them and which is included in A.

Lemma 14.1. *Let A be a non-empty connected open set in the Euclidean space $\mathbb{R}^2$. Then A is path-connected.*

Proof. Choose a point $\mathbf{x} \in A$. Define the set $O_1 \subset A$ which contains all the points $\mathbf{y} \in A$ which can be connected with $\mathbf{x}$ by an arc. Let us show that O_1 is open. If $\mathbf{y} \in O_1$ then there exists an arc $\gamma([a, b]) \subset A$ with $\gamma(a) = \mathbf{x}$ and $\gamma(b) = \mathbf{y}$. Since $\mathbf{y}$ is an interior point of A, there exists $\epsilon > 0$ such that if $||\mathbf{x}' - \mathbf{y}|| < \epsilon$ then $\mathbf{x}' \in A$. But all such points can be joined with $\mathbf{y}$ by a straight line included in A. Thus $\mathbf{x}$ can be joined with $\mathbf{x}'$ by an arc included in A, hence $B_\epsilon(\mathbf{y}) \subset O_1$.

Now if $O_1 = A$ then we are done. If not, the set $O_2 := A \setminus O_1$ is not empty. One can prove in a similar manner that O_2 is open: if $\mathbf{y}$ cannot be joined with $\mathbf{x}$ by an arc included in A, then no points close enough to $\mathbf{y}$ can be joined with $\mathbf{x}$. But since $A = O_1 \cup O_2$, it would mean that A is not connected, contradiction. □

Lemma 14.2. *Let M be a compact set in the Euclidean space $\mathbb{R}^2$. If $\mathbf{x} \in \mathbb{R}^2$ we define*

$$\mathbb{R}^2 \ni \mathbf{x} \mapsto d(\mathbf{x}, M) := \inf\{||\mathbf{y} - x|| : \ \mathbf{y} \in M\} \in \mathbb{R}.$$

Then this map is continuous.

Proof. For a fixed $\mathbf{x}$, the map $M \ni \mathbf{y} \mapsto ||\mathbf{y} - \mathbf{x}|| \in \mathbb{R}$ is continuous and defined on a compact set. Thus it attains its minimum at some point $\mathbf{y}_{\mathbf{x}} \in M$. Hence for every $\mathbf{x} \in \mathbb{R}^2$ there exists $\mathbf{y}_x \in M$ such that $d(\mathbf{x}, M) = ||\mathbf{x} - \mathbf{y}_{\mathbf{x}}||$. We note the inequalities:

$$d(\mathbf{x}', M) \leq ||\mathbf{x}' - \mathbf{y}_x|| \leq ||\mathbf{x}' - \mathbf{x}|| + ||\mathbf{x} - \mathbf{y}_{\mathbf{x}}|| = ||\mathbf{x}' - \mathbf{x}|| + d(\mathbf{x}, M),$$

which due to the symmetry lead to:

$$|d(\mathbf{x}, M) - d(\mathbf{x}', M)| \leq ||\mathbf{x} - \mathbf{x}'||$$

for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$, which proves Lipschitz continuity. □

The next lemma is a poor man's version of the Tietze extension theorem. It roughly says that given an arc M in $\mathbb{R}^2$ contained in a large closed ball B, we can find a continuous function $g : B \mapsto M$ which extends the identity map on M.

Lemma 14.3. *Let M be an arc in $\mathbb{R}^2$. Let $\mathbf{x}_0 \in \mathbb{R}^2$ and consider $B_r(\mathbf{x}_0)$ a ball sufficiently large such that $M \subset B_r(\mathbf{x}_0)$. Then there exists a continuous map $g : \overline{B_r(\mathbf{x}_0)} \mapsto M$ such that $g(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in M$.*

Proof. Since M is an arc, it is bounded and closed, thus compact. Moreover, M is homeomorphic with a closed interval in $\mathbb{R}$, thus we can find a continuous function $\gamma : [0, 1] \mapsto M$ with continuous inverse $\gamma^{-1} : M \mapsto [0, 1]$. If we can find a continuous function $F : \overline{B_r(\mathbf{x}_0)} \mapsto [0, 1]$ such that $F(\mathbf{x}) = \gamma^{-1}(\mathbf{x})$ for all $\mathbf{x} \in M$, then the extension we are looking for is $g = \gamma \circ F$.

Let us define the function F. If $\mathbf{x} \in M$ we put $F(\mathbf{x}) = \gamma^{-1}(\mathbf{x})$. If $\mathbf{x} \in \overline{B_r(\mathbf{x}_0)} \setminus M$ we put

$$F(\mathbf{x}) := \inf_{\mathbf{y} \in M} \left\{ \gamma^{-1}(\mathbf{y}) + \frac{||\mathbf{x} - \mathbf{y}||}{d(\mathbf{x}, M)} - 1 \right\}.$$

If $\mathbf{x} \notin M$, the map

$$M \ni \mathbf{y} \mapsto \gamma^{-1}(\mathbf{y}) + \frac{||\mathbf{x} - \mathbf{y}||}{d(\mathbf{x}, M)} - 1 \in \mathbb{R}$$

is continuous and defined on a compact set, thus there exists some $\mathbf{w}(\mathbf{x}) \in M$ such that

$$F(\mathbf{x}) = \gamma^{-1}(\mathbf{w}(\mathbf{x})) + \frac{||\mathbf{x} - \mathbf{w}(\mathbf{x})||}{d(\mathbf{x}, M)} - 1. \tag{14.2}$$

Let us show that the range of F is the interval $[0, 1]$. If $\mathbf{x} \in M$ it is obvious. If $\mathbf{x} \notin M$ then we know from Lemma 14.2 that there exists some $\mathbf{y}_{\mathbf{x}} \in M$ such that $0 < ||\mathbf{x} - \mathbf{y}_{\mathbf{x}}|| = d(\mathbf{x}, M) \leq ||\mathbf{x} - \mathbf{y}||$ for all $\mathbf{y} \in M$. Thus $0 \leq \gamma^{-1}(\mathbf{y}) + \frac{||\mathbf{x} - \mathbf{y}||}{d(\mathbf{x}, M)} - 1$, for all $\mathbf{y} \in M$, which implies that $0 \leq F(\mathbf{x})$. Moreover,

$$F(\mathbf{x}) \leq \gamma^{-1}(\mathbf{y}_{\mathbf{x}}) + \frac{||\mathbf{x} - \mathbf{y}_{\mathbf{x}}||}{d(\mathbf{x}, M)} - 1 = \gamma^{-1}(\mathbf{y}_{\mathbf{x}}) \leq 1.$$

Now we want to prove that F is continuous. Let $\mathbf{a} \in M$. According to the definition of F, we have that $F(\mathbf{a}) = \gamma^{-1}(\mathbf{a})$. Consider any sequence $\{\mathbf{x}_n\} \subset \overline{B_r(\mathbf{x}_0)}$ which converges to $\mathbf{a}$. We can split it in a subsequence included in M, denoted by $\{\mathbf{x}_n^M\}$, and a subsequence in $\overline{B_r(\mathbf{x}_0)} \setminus M$, denoted by $\{\mathbf{x}_n^{M^c}\}$. Since $F(\mathbf{x}_n^M) = \gamma^{-1}(\mathbf{x}_n^M)$ and because γ^{-1} is continuous on M, we have $F(\mathbf{x}_n^M) \to F(\mathbf{a}) = \gamma^{-1}(\mathbf{a})$. What we have to prove now is that $F(\mathbf{x}_n^{M^c}) \to \gamma^{-1}(\mathbf{a})$ provided $\mathbf{x}_n^{M^c} \to \mathbf{a}$. We note that the sequence of minimizing points $\mathbf{w}(\mathbf{x}_n^{M^c})$ must converge to $\mathbf{a}$; otherwise, since $d(\mathbf{x}_n^{M^c}, M) \leq ||\mathbf{x}_n^{M^c} - \mathbf{a}|| \to 0$, we would eventually have $F(\mathbf{x}_{n_k}^{M^c}) > 1$ for some subsequence. Moreover, any sequence $\mathbf{y}_{\mathbf{x}_n^{M^c}}$ defined by $d(\mathbf{x}_n^{M^c}, M) = ||\mathbf{x}_n^{M^c} - \mathbf{y}_{\mathbf{x}_n^{M^c}}|| \leq ||\mathbf{x}_n^{M^c} - \mathbf{a}||$ must converge to $\mathbf{a}$. Then we have:

$$\gamma^{-1}(\mathbf{w}(\mathbf{x}_n^{M^c})) \leq F(\mathbf{x}_n^{M^c}) \leq \gamma^{-1}(\mathbf{y}_{\mathbf{x}_n^{M^c}}),$$

where the first inequality is a consequence of (14.2) and of $||\mathbf{x}_n^{M^c} - \mathbf{w}(\mathbf{x}_n^{M^c})|| \geq d(\mathbf{x}_n^{M^c}, M)$, while the second inequality is a consequence of the definition of F. Then the continuity of γ^{-1} at $\mathbf{a}$ ensures that $F(\mathbf{x}_n^{M^c}) \to \gamma^{-1}(\mathbf{a})$ and we are done.

Now let $\mathbf{a} \in \overline{B_r(\mathbf{x}_0)} \setminus M$ and consider any sequence $\mathbf{x}_n \to \mathbf{a}$. Since M is closed, we may consider that $\mathbf{x}_n \notin M$ for all n. Then we can write:

$$F(\mathbf{a}) = \gamma^{-1}(\mathbf{w}(\mathbf{a})) + \frac{||\mathbf{a} - \mathbf{w}(\mathbf{a})||}{d(\mathbf{a}, M)} - 1, \qquad F(\mathbf{x}_n) = \gamma^{-1}(\mathbf{w}(\mathbf{x}_n)) + \frac{||\mathbf{x}_n - \mathbf{w}(\mathbf{x}_n)||}{d(\mathbf{x}_n, M)} - 1.$$

From the definition of F and using the triangle inequality we have:

$$\begin{aligned} F(\mathbf{a}) &\leq \gamma^{-1}(\mathbf{w}(\mathbf{x}_n)) + \frac{||\mathbf{a} - \mathbf{w}(\mathbf{x}_n)||}{d(\mathbf{a}, M)} - 1 \\ &\leq F(\mathbf{x}_n) + \frac{||\mathbf{a} - \mathbf{x}_n||}{d(\mathbf{a}, M)} + \left(\frac{1}{d(\mathbf{a}, M)} - \frac{1}{d(\mathbf{x}_n, M)} \right) ||\mathbf{w}(\mathbf{x}_n) - \mathbf{x}_n||. \end{aligned}$$

Using the continuity of the distance $d(\cdot, M)$ from Lemma 14.2, we obtain $F(\mathbf{a}) \leq \liminf F(\mathbf{x}_n)$.

In the same way we have:

$$F(\mathbf{x}_n) \leq \gamma^{-1}(\mathbf{w}(\mathbf{a})) + \frac{||\mathbf{x}_n - \mathbf{w}(\mathbf{a})||}{d(\mathbf{x}_n, M)} - 1 \leq F(\mathbf{a}) + \frac{||\mathbf{x}_n - \mathbf{a}||}{d(\mathbf{x}_n, M)} + \left(\frac{1}{d(\mathbf{x}_n, M)} - \frac{1}{d(\mathbf{a}, M)} \right) ||\mathbf{w}(\mathbf{a}) - \mathbf{a}||$$

hence $\limsup F(\mathbf{x}_n) \leq F(\mathbf{a})$. Thus F is continuous and the proof is over.

□

The following lemma has a quite obvious 'proof by drawing', but it's rigorous argument is based on Brouwer's fixed point theorem.

Lemma 14.4. *Let K be the rectangle $\{[x, y] : a \leq x \leq b, c \leq y \leq d\} \subset \mathbb{R}^2$. Assume that we have two arcs $\gamma, \phi : [-1, 1] \mapsto K$, $j \in \{1, 2\}$, such that $\gamma(-1)$ belongs to the left side $\{[a, y] : y \in [c, d]\}$ of K, $\gamma(1)$ belongs to the right side $\{[b, y] : y \in [c, d]\}$ of K, $\phi(-1)$ belongs to the upper side $\{[x, d] : x \in [a, b]\}$ of K, and $\phi(1)$ belongs to the lower side $\{[x, c] : x \in [a, b]\}$ of K. Then the two arcs must cross each other, i.e. there exist $s, t \in [-1, 1]$ such that $\gamma(t) = \phi(s)$.*

Proof. Denote by $\gamma(t) = [x(t), y(t)]$ and by $\phi(s) = [u(s), w(s)]$. Assume that the two arcs never cross. It means that the quantity:

$$N(t, s) := \max\{|x(t) - u(s)|, \ |y(t) - w(s)|\}, \quad [t, s] \in [-1, 1] \times [-1, 1]$$

is strictly positive. By the triangle inequality:

$$|x(t) - u(s)| - |x(t_0) - u(s_0)| \leq |x(t) - x(t_0) - u(s) + u(s_0)| \leq |x(t) - x(t_0)| + |u(s) - u(s_0)|,$$

or

$$|x(t) - u(s)| \leq |x(t_0) - u(s_0)| + \epsilon \leq N(t_0, s_0) + \epsilon$$

if $[t, s]$ is close enough to $[t_0, s_0]$, due to the continuity of x and u. In a similar way we can prove $|y(t) - w(s)| \leq N(t_0, s_0) + \epsilon$, thus by taking the maximum we obtain $N(t, s) \leq N(t_0, s_0) + \epsilon$. By symmetry we must also have the inequality $N(t_0, s_0) \leq N(t, s) + \epsilon$ hence N is continuous.

Since N is continuous, positive and defined on a compact set, it must have a positive minimum. Thus $1/N(t, s)$ is also continuous on $[-1, 1] \times [-1, 1]$. Define:

$$f : [-1, 1] \times [-1, 1] \mapsto [-1, 1] \times [-1, 1], \quad f(t, s) := \left[-\frac{x(t) - u(s)}{N(t, s)}, -\frac{y(t) - w(s)}{N(t, s)} \right].$$

Due to our assumptions, $x(-1) = a$, $w(-1) = d$, $x(1) = b$ and $w(1) = c$. The function f is continuous and defined on a convex body. According to Brouwer's fixed point theorem, it must have a fixed point. Note that the range of f belongs in fact to the boundary of the square. Thus if $f([t_0, s_0]) = [t_0, s_0]$ is a fixed point of f, we must either have $|t_0| = 1$ or $|s_0| = 1$.

If $t_0 = 1$, then we would have $1 = -\frac{x(1)-u(s_0)}{N(1,s_0)} = -\frac{b-u(s_0)}{N(1,s_0)} \leq 0$, impossible. If $t_0 = -1$ we would have $-1 = -\frac{x(-1)-u(s_0)}{N(-1,s_0)} = -\frac{a-u(s_0)}{N(-1,s_0)} \geq 0$, again impossible.

If $s_0 = 1$ we would have $1 = -\frac{y(t_0)-w(1)}{N(t_0,1)} = -\frac{y(t_0)-c}{N(t_0,1)} \leq 0$, impossible. If $s_0 = -1$ we would have $-1 = -\frac{y(t_0)-w(-1)}{N(t_0,-1)} = -\frac{y(t_0)-d}{N(t_0,-1)} \geq 0$, again impossible.

Thus f cannot have fixed points, which shows that our assumption on the positivity of N was false.

□

The next lemma says that if an arc starts inside a closed rectangle and ends outside it, then it must cross the boundary.

Lemma 14.5. *Let K be the rectangle $\{[x, y] : \ a \leq x \leq b \text{ and } c \leq y \leq d\} \subset \mathbb{R}^2$. Assume that $\gamma : [0, 1] \mapsto \mathbb{R}^2$ is an arc such that $\gamma(0) \in \mathrm{Int}(K)$ and $\gamma(1) \notin K$. Then there exists $0 < c < 1$ such that $\gamma(c) \in \partial K$ and $\gamma(t) \in \mathrm{Int}(K)$ for all $0 \leq t < c$.*

Proof. We start by noting that K is closed, the interior of K is given by

$$\mathrm{Int}(K) = \{[x, y] : \ a < x < b \text{ and } c < y < d\},$$

the exterior of K is

$$K^c = \{[x, y] : \ x < a \text{ or } b < x \text{ or } y < c \text{ or } d < y\},$$

and the boundary is $\partial K = K \setminus \mathrm{Int}(K)$.

Denote by A the subset of the interval $[0, 1] \subset \mathbb{R}$ defined by:

$$A := \{0 \leq t \leq 1 : \gamma(s) \in \mathrm{Int}(K), \ \forall 0 \leq s \leq t\}.$$

In words, if $t \in A$, then all the points of the arc γ corresponding to previous parameter values $s \leq t$ lie in the open rectangle. Since A is bounded, it has a supremum which we denote by c. Denote by $[x(t), y(t)] := \gamma(t)$. Since $a < x(0) < b$ and $c < y(0) < d$, and because x and y are continuous functions, the previous strict inequalities will remain true in a neighborhood of 0. This shows that $c > 0$. Moreover, since c is the supremum of A, there exists a sequence $\{t_n\}_{n\geq 1} \subset A$ such that $\lim_{n\to\infty} t_n = c$. This means in particular that $a < x(t_n) < b$ and $c < y(t_n) < d$ for all n, and by taking the limit using the continuity of x and y at c we obtain $a \leq x(c) \leq b$ and $c \leq y(c) \leq d$. In other words, $\gamma(s) \in \mathrm{Int}(K)$ if $0 \leq s < c$ and $\gamma(c) \in K$.

Assume without loss of generality that $\gamma(1) = [\xi_1, \xi_2] \in K^c$ with $\xi_1 < a$. Since $a \leq x(c)$, we have that $c < 1$. Moreover, there exists N large enough such that $c + 1/n < 1$ for all $n \geq N$. Because $c + 1/n$ is not an element of A, we may find some $0 \leq s_n \leq c + 1/n$ such that $\gamma(s_n) := [u_n, w_n] \notin \mathrm{Int}(K)$. Moreover, we must have $c \leq s_n$ because we know that for all $t < c$ we have $\gamma(t) \in \mathrm{Int}(K)$. Thus $c \leq s_n \leq c+1/n$. In other words, $s_n \to c$ and $\gamma(s_n) = [u_n, w_n] \notin \mathrm{Int}(K)$. Hence at least one of the following four possibilities must occur: $u_n \leq a$ or $u_n \geq b$ or $w_n \leq c$ or $w_n \geq d$. There must exist a subsequence $\{s_{n_k}\}_{k\geq 1}$ such that exactly one of the above four inequalities is satisfied for all $k \geq 1$. Without loss of generality, assume that $u_{n_k} \geq b$ for all $k \geq 1$. Since $s_{n_k} \to c$, due to the continuity of γ at c we must have that the first component of $\gamma(c)$ must obey the same inequality as the first component of $\gamma(s_{n_k})$. Thus $\gamma(c) \notin \mathrm{Int}(K)$. But we proved before that $\gamma(c) \in K$. Thus $\gamma(c) \in K \setminus \mathrm{Int}(K) = \partial K$ and we are done.

□

14.2 The main theorem

If $\phi : S^1 \mapsto \mathbb{R}^2$ is a homeomorphism which maps the unit circle into the plane, then the image $J := \phi(S^1)$ is called a Jordan curve. In other words, a Jordan curve is a simple closed path which can be parametrized by

$$[0, 2\pi] \ni t \mapsto \phi(\cos(t), \sin(t)) \in \mathbb{R}^2.$$

A Jordan curve is bounded and closed, thus compact.

Theorem 14.6. *(Jordan curve theorem). Let J be any Jordan curve. Then the set $\mathbb{R}^2 \setminus J$ is open in $\mathbb{R}^2$, has exactly two connected components (one bounded and the other one unbounded), and J is their boundary.*

Proof. We start by proving that $\mathbb{R}^2 \setminus J$ is not connected, i.e. it has at least two connected components. Assume that $\mathbb{R}^2 \setminus J$ is connected. According to Lemma 14.1, since $\mathbb{R}^2 \setminus J$ is open (because J is closed), it must be path connected. The strategy is to construct an 'inner' point $\mathbf{x}_i$ which cannot be joined with the points situated outside some large ball which contains J.

Let us construct this point $\mathbf{x}_i$. The map

$$\mathbb{R}^4 \supset J \times J \ni [\mathbf{x}, \mathbf{y}] \mapsto ||\mathbf{x} - \mathbf{y}|| \in \mathbb{R}$$

is continuous and defined on a compact set. Thus there exist $\mathbf{x}_l$ and $\mathbf{x}_r$ in J which maximize this distance, i.e. $||\mathbf{x} - \mathbf{y}|| \leq ||\mathbf{x}_r - \mathbf{x}_l||$ for all $\mathbf{x}, \mathbf{y} \in J$. Denote by $\overline{\mathbf{x}_r\mathbf{x}_l}$ the straight segment joining the two points, and consider the two infinite lines L_l and L_r which are perpendicular on $\overline{\mathbf{x}_r\mathbf{x}_l}$ and pass through $\mathbf{x}_l$ and $\mathbf{x}_r$ respectively. No point of L_l other than $\mathbf{x}_l$, and no point of L_r other than $\mathbf{x}_r$ can belong to J, otherwise $||\mathbf{x}_r - \mathbf{x}_l||$ would not be maximal. Thus J belongs to the strip generated by the two lines. Moreover, because J is bounded, we can build a closed rectangle K which includes J in its interior and has two parallel sides included in L_l and L_r.

Without loss of generality, we may assume that $\mathbf{x}_l = [-1, 0]$, $\mathbf{x}_r = [1, 0]$ and $K = \{[x, y] : -1 \leq x \leq 1,\ -10 \leq y \leq 10\}$. The curve J has exactly two points in common with K, and they are $\mathbf{x}_r$ and $\mathbf{x}_l$. These two points split J into two arcs: J_u and J_d. Without loss of generality we may assume that J_u starts at $\mathbf{x}_r$ and ends at $\mathbf{x}_l$ with the trigonometric orientation, while J_d starts at $\mathbf{x}_l$ and ends at $\mathbf{x}_r$ with the same trigonometric orientation.

The segment linking the top point $T = [0, 10]$ with the bottom point $B = [0, -10]$ is denoted by $\overline{TB}$. We note that J_u and $\overline{TB}$ are two arcs which must cross at least in one point, due to Lemma 14.4. Denote by $\mathbf{y}_{max,u}$ the point of $J_u \cap \overline{TB}$ with the largest second coordinate, i.e. the crossing point closest to T. Denote by $\mathbf{y}_{min,u}$ the point of $J_u \cap \overline{TB}$ with the lowest second coordinate, i.e. the crossing point closest to B. Note that it can happen that $\mathbf{y}_{max,u} = \mathbf{y}_{min,u}$.

In the same way, the arc J_d and the segment $\overline{TB}$ must cross. Denote by $\mathbf{y}_{max,d}$ the point of $J_d \cap \overline{TB}$ with the largest second coordinate, i.e. the crossing point closest to T. Denote by $\mathbf{y}_{min,d}$ the point of $J_d \cap \overline{TB}$ with the smallest second coordinate, i.e. the crossing point closest to B.

Define the 'inner' point which we talked about to be $\mathbf{x}_i := (\mathbf{y}_{min,u} + \mathbf{y}_{max,d})/2$. Clearly, $\mathbf{x}_i$ is not an element of J and belongs to $\mathbb{R}^2 \setminus J$. If $\mathbb{R}^2 \setminus J$ were connected, we can join $\mathbf{x}_i$ with any other point from outside K, since $K^c \subset \mathbb{R}^2 \setminus J$. According to Lemma 14.5, such an arc must cross the boundary of the rectangle K in some point $\mathbf{w}$. This $\mathbf{w}$ can be neither $\mathbf{x}_r$ nor $\mathbf{x}_l$, since they belong to J.

If the second coordinate of $\mathbf{w}$ is negative, then consider the arc starting at T, continued with a straight segment to $\mathbf{y}_{max,u}$, continued with the part of J_u between $\mathbf{y}_{max,u}$ and $\mathbf{y}_{min,u}$, then by the straight segment to $\mathbf{x}_i$, then by the arc linking $\mathbf{x}_i$ with $\mathbf{w}$, and then we continue on the boundary of K until we reach B. In this way we constructed an arc in K starting at T and ending at B which has no common points with J_d, contradicting Lemma 14.4.

If the second coordinate of $\mathbf{w}$ is positive, then consider the arc starting at B, continued with a straight segment to $\mathbf{y}_{min,d}$, then continued with the part of J_d between $\mathbf{y}_{min,d}$ and $\mathbf{y}_{max,d}$, then with a straight segment to $\mathbf{x}_i$, then with the arc from $\mathbf{x}_i$ to $\mathbf{w}$, and then on the boundary of K until we reach T. In this way we constructed an arc in K linking B with T which does not cross J_u, again a contradiction. Thus $\mathbb{R}^2 \setminus J$ is not connected.

Up to now we know that there exists exactly one unbounded connected component (which contains K^c), and at least one bounded 'inner' component. The next result is about the boundary of each such connected component: it says that if U is a connected component of $\mathbb{R}^2 \setminus J$, then U is open in $\mathbb{R}^2$ and the boundary $\partial U = \overline{U} \setminus U$ equals J.

The first observation is that $\partial U \subset J$; if this was not true, then there exists some point $\mathbf{x}$ in $\overline{U}$ which belongs neither to U nor to J, hence it must be an element of some other connected component W. But then $\mathbf{x}$ is an inner point of W and must be isolated from U, contradiction with our assumption that $\mathbf{x}$ belongs to the closure of U.

It could happen though that ∂U is strictly included in but not equal with J. In this case, there exists an arc $M \subset J$ such that $\partial U \subset M$. We will show that this leads to a contradiction.

We first assume that U is a bounded connected component. Consider a closed ball $D := \overline{B_R(\mathbf{x}_o)}$ where $\mathbf{x}_o$ is some inner point of U, and $R > 0$ is sufficiently large such that the circle ∂D belongs to K^c, thus outside U. Clearly, $M \subset \overline{U} \subset D$. According to Lemma 14.3, there exists a continuous map $g : D \mapsto M$ such that $g(\mathbf{x}) = \mathbf{x}$ on M. Define the map $q : D \mapsto D \setminus \{\mathbf{x}_o\}$ given by $q(\mathbf{x}) = g(\mathbf{x}) \in M$ if $\mathbf{x} \in \overline{U}$ and $q(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in D \setminus U$. Note that q is well defined because the 'dangerous' points of $\overline{U} \cap (D \setminus U)$ are included in M where $g(\mathbf{x}) = \mathbf{x}$. Moreover, q is continuous, and $\mathbf{x}_o$ is never in its range. Let $t : D \setminus \{\mathbf{x}_o\} \mapsto \partial D$ be the natural retraction, i.e. the map which sends $\mathbf{x} \in D \setminus \{\mathbf{x}_o\}$ into the point on ∂D obtained from the intersection of ∂D with the ray starting from $\mathbf{x}_o$ and going through $\mathbf{x}$. Let $a : \partial D \mapsto \partial D$ be the antipodal map, i.e. the map which sends a point of ∂D into the diametrically oposed point. Now define the map $r : D \mapsto D$ given by $r = a \circ t \circ q$. We note that r is continuous, and its range is ∂D. Brouwer's fixed point theorem says that r must have a fixed point, which can only be on the boundary ∂D. But $q(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in \partial D \subset U^c$, and $t(\mathbf{x}) = \mathbf{x}$ on the boundary. The antipodal map a prohibits the existence of a fixed point on the boundary for r, which leads to a contradiction. Thus $\partial U = J$ if U is bounded.

If W is the unbounded connected component and M is an arc containing the boundary of W, then we can use the point $\mathbf{x}_o$ and the disk D previously considered in order to define $q : D \mapsto D \setminus \{\mathbf{x}_o\}$ by $q(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in \overline{W}$ and $q(\mathbf{x}) = g(\mathbf{x}) \in M$ if $\mathbf{x} \in D \setminus W$. From here the rest of the argument is identical, and leads to a contradiction. Thus $\partial W = J$.

Let us recapitulate what we know until now: there is exactly one unbounded component, at least one bounded, and every connected component has its boundary equal with J. The last thing we have to prove is that there are no other bounded components besides the component U_i which contains the point $\mathbf{x}_i$ defined when we proved that $\mathbb{R}^2 \setminus J$ is not connected.

Now let us assume that there exists a bounded component $W \neq U_i$. The points $\mathbf{x}_r$ and $\mathbf{x}_l$ belong to the boundary of W, hence both of them are limits of sequences of points from W. In particular, there exists $\tilde{\mathbf{x}}_r \in W$ such that its first component is larger than $1/2$, and there exists $\tilde{\mathbf{x}}_l \in W$ such that its first component is smaller than $-1/2$. Because W is path connected, there exists a path $\widehat{\tilde{\mathbf{x}}_l\tilde{\mathbf{x}}_r} \subset W$.

Now consider the path $\widehat{TB}$ starting from T, continued with the straight segment joining T with $\mathbf{y}_{max,u}$, continued with the arc of J between $\mathbf{y}_{max,u}$ and $\mathbf{y}_{min,u}$, then with the straight segment (containing $\mathbf{x}_i$) between $\mathbf{y}_{min,u}$ and $\mathbf{y}_{max,d}$, continued with the arc of J between $\mathbf{y}_{max,d}$ and $\mathbf{y}_{min,d}$, and finally continued with the straight segment between $\mathbf{y}_{min,d}$ and B. We see that all the points of $\widehat{TB}$ belong either to J, to U_i or to the unbounded connected component. It means that $\widehat{TB}$ and $\widehat{\tilde{\mathbf{x}}_l\tilde{\mathbf{x}}_r} \subset W$ cannot have common points, and this contradicts Lemma 14.4.

$\square$