Exercises in ordinary differential equations

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Exercise 1 (Grönwall inequality). Consider a non-negative continuous function $f : (a, b) \mapsto \mathbb{R}$ and fix $t_0 \in (a, b)$. Assume that the following inequality holds for every $t \in (a, b)$, where C and α are positive constants:

$$f(t) \leq \alpha + C \left| \int_{t_0}^{t} f(s) ds \right|.$$

Show that $0 \leq f(t) \leq \alpha e^{C|t-t_0|}$.

Hint. Assume that $t > t_0$. In this case we have:

$$f(t) \leq \alpha + C \int_{t_0}^{t} f(s) ds.$$

We may rewrite this inequality in the following way:

$$f(s) \leq \alpha + C \int_{t_0}^{s} f(t_1) dt_1.$$

Introducing the second inequality into the first one we get:

$$f(t) \leq \alpha + C(t - t_0)\alpha + C^2 \int_{t_0}^{t} \left(\int_{t_0}^{s} f(t_1) dt_1 \right) ds.$$

Using

$$f(t_1) \leq \alpha + C \int_{t_0}^{t_1} f(t_2) dt_2$$

in the previous inequality we obtain:

$$f(t) \leq \alpha + C(t - t_0)\alpha + \frac{C^2(t - t_0)^2}{2}\alpha + C^3 \int_{t_0}^{t} \left(\int_{t_0}^{s} \left(\int_{t_0}^{t_1} f(t_2) dt_2 \right) dt_1 \right) ds$$

where we used the identity:

$$\int_{t_0}^{t} \left(\int_{t_0}^{s} \alpha dt_1 \right) ds = \frac{(t - t_0)^2}{2}\alpha.$$

In general,

$$\int_{t_0}^{t} \left(\int_{t_0}^{s} \left(\int_{t_0}^{t_1} \dots \left(\int_{t_0}^{t_{n-1}} \alpha dt_n \right) \dots \right) dt_1 \right) ds = \frac{(t - t_0)^{n+1}}{(n + 1)!}\alpha, \quad \forall n \geq 1.$$

By repeating the iteration, we get for every $n \geq 2$ that:

$$f(t) \leq \alpha \sum_{j=0}^{n} \frac{C^j (t - t_0)^j}{j!} + C^{n+1} \int_{t_0}^{t} \left(\int_{t_0}^{s} \left(\int_{t_0}^{t_1} \dots \left(\int_{t_0}^{t_{n-1}} f(t_n) dt_n \right) \dots \right) dt_1 \right) ds. \tag{0.1}$$

The function f is continuous on the compact interval $[t_0, t]$, thus it has a maximum $M < \infty$. Since $0 \leq f(x) \leq M$ for all $t_0 \leq x \leq t$, by replacing $f(t_n)$ with M in (0.1) and integrating we obtain:

$$f(t) \le \alpha \sum_{j=0}^{n} \frac{C^j (t-t_0)^j}{j!} + M\frac{C^{n+1}(t-t_0)^{n+1}}{(n+1)!}, \quad \forall n \ge 2.$$

We observe that $\frac{C^{n+1}(t-t_0)^{n+1}}{(n+1)!}$ converges to 0 when n goes to infinity (why?), while $\sum_{j=0}^{n} \frac{C^j (t-t_0)^j}{j!}$ converges to $e^{C(t-t_0)}$ and we are done.

If $t < t_0$ the argument is similar, based on the inequality

$$f(t) \le \alpha + C\int_t^{t_0} f(s)ds.$$

Exercise 2. Consider a function $\mathbf{f} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ where $\mathbf{f} \in C^1(\mathbb{R}^{d+1})$, which obeys the estimate

$$||\mathbf{f}(t, \mathbf{x})|| \le C||\mathbf{x}||, \quad \forall [t, \mathbf{x}] \in \mathbb{R}^{d+1}.$$

Consider the equation $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$, where $\mathbf{y}(0) = \mathbf{y}_0$. Show that there exists a **unique** solution $\mathbf{y} : \mathbb{R} \mapsto \mathbb{R}^d$ which solves the equation **for all** $t \in \mathbb{R}$.

Hint. Lemma 7.3 in my Analysis notes implies that $\mathbf{f}$ obeys a local Lipschitz condition on the set $[-1, 1] \times \overline{B_1(\mathbf{y}_0)}$. Then Theorem 6.3 in my notes (local existence) states that there exists a positive $\delta_1 > 0$ and a differentiable function $\mathbf{y} : (-\delta_1, \delta_1) \mapsto \mathbb{R}^d$ which is a **local** solution to our ODE which also obeys:

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s, \mathbf{y}(s))ds, \quad |t| < \delta_1.$$

The hard part of this exercise is to show that the above solution exists for all t. This is what we do now.

Using the estimate $||\mathbf{f}(s, \mathbf{y}(s))|| \le C||\mathbf{y}(s)||$ we can write (take $t > 0$):

$$||\mathbf{y}(t)|| \le ||\mathbf{y}(0)|| + C\int_0^t ||\mathbf{y}(s)||ds.$$

We know that the function $||\mathbf{y}(t)||$ is continuous, hence Exercise 1 (Grönwall inequality) gives:

$$||\mathbf{y}(t)|| \le ||\mathbf{y}(0)||e^{C|t|}.$$

Now assume that we cannot find a global in time solution, i.e. it only exists for a time interval of the form $(-T_1, T_2)$ where $T := \min\{T_1, T_2\} < \infty$ and $0 < \delta_1 < T$. Assume without loss of generality that $T = T_2 < \infty$. We then have:

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s, \mathbf{y}(s))ds, \quad ||\mathbf{y}(t)|| \le ||\mathbf{y}(0)||e^{CT}, \quad |t| < T.$$

Consider an arbitrary sequence $\{t_m\}_{m\ge1} \subset (0, T)$ which converges to T. Being convergent, the sequence $\{t_m\}_{m\ge1}$ is Cauchy. Let us show that the sequence of values $\{\mathbf{y}(t_m)\}_{m\ge1} \subset \mathbb{R}^d$ is also Cauchy. If $p > q$ we can write (assume without loss that $t_p > t_q$):

$$||\mathbf{y}(t_p) - \mathbf{y}(t_q)|| = ||\int_{t_q}^{t_p} \mathbf{f}(s, \mathbf{y}(s))ds|| \le \int_{t_q}^{t_p} ||\mathbf{f}(s, \mathbf{y}(s))||ds \le C||\mathbf{y}(0)||e^{CT}|t_p - t_q|$$

which can be made arbitrarily small since $\{t_m\}_{m\ge1}$ is Cauchy. Hence $\{\mathbf{y}(t_n)\}_{n\ge1}$ converges to some vector $\mathbf{y}_T \in \mathbb{R}^d$. It $\{s_n\}_{n\ge1}$ is some other sequence in $(0, T)$ which converges to T, we have:

$$||\mathbf{y}(s_n) - \mathbf{y}(t_n)|| \le C||\mathbf{y}(0)||e^{CT}|s_n - t_n| \to 0$$

which shows that the limit $\mathbf{y}_T$ is independent of the sequence we choose. Thus we have:

$$\lim_{t \nearrow T} \mathbf{y}(t) =: \mathbf{y}(T-0) = \mathbf{y}_T, \quad \lim_{t \nearrow T} \mathbf{y}'(t) =: \mathbf{y}'(T-0) = \mathbf{f}(T, \mathbf{y}_T).$$

Now consider the initial value problem $\tilde{\mathbf{y}}'(t) = \mathbf{f}(t, \tilde{\mathbf{y}}(t))$, where $\tilde{\mathbf{y}}(T) = \mathbf{y}_T$. The same local existence Theorem 6.3 (where we put $t_0 = T$ and $\mathbf{y}_0 = \mathbf{y}_T$) allows us to construct a solution on an interval $(T - \delta_2, T + \delta_2)$ and we have:

$$\lim_{t \searrow T} \tilde{\mathbf{y}}'(t) =: \tilde{\mathbf{y}}'(T+0) = \tilde{\mathbf{y}}'(T) = \mathbf{f}(T, \mathbf{y}_T).$$

Now we can define a function $\mathbf{z} : (-T_1, T_2 + \delta_2)$ where $\mathbf{z}(t) = \mathbf{y}(t)$ on $(-T_1, T_2)$ and $\mathbf{z}(t) = \tilde{\mathbf{y}}(t)$ on $[T_2, T_2 + \delta_2)$. We observe that $\mathbf{z}$ is continuously differentiable and solves the ODE. Thus T_2 can be made larger, which provides a contradiction.

Concerning uniqueness: assume that there exist two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ which both solve the differential equation and $\mathbf{y}_1(0) = \mathbf{y}_2(0) = \mathbf{y}_0$. We already know that they exist for all t. Both of them obey the bound $||\mathbf{y}_j(t)|| \leq ||\mathbf{y}_0|| e^{C|t|}$. If $\mathbf{y}_0 = 0$ then both of them are identically zero (thus equal). Hence we may assume that $\mathbf{y}_0 \neq 0$.

Fix some $T > 0$. If $|t| \leq T$, then both vectors $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ will be contained in the closed ball $\overline{B_R(\mathbf{y}_0)}$ with $R := ||\mathbf{y}_0|| e^{CT}$.

We know from Lemma 7.3 that there exists some $L < \infty$ such that:

$$||\mathbf{f}(s, \mathbf{x}) - \mathbf{f}(s, \mathbf{z})|| \leq L ||\mathbf{x} - \mathbf{z}||, \quad \forall |s| \leq T, \quad \forall \mathbf{x}, \mathbf{z} \in \overline{B_R(\mathbf{y}_0)}.$$

We have the identity:

$$\mathbf{y}_2(t) - \mathbf{y}_1(t) = \int_0^t [\mathbf{f}(s, \mathbf{y}_2(s)) - \mathbf{f}(s, \mathbf{y}_1(s))] ds, \quad \forall |t| \leq T.$$

Let $h(t) := ||\mathbf{y}_2(t) - \mathbf{y}_1(t)||$, with $h(0) = 0$. Assume that $t > 0$. Reasoning as before, we can write:

$$0 \leq h(t) \leq \int_0^t ||\mathbf{f}(s, \mathbf{y}_2(s)) - \mathbf{f}(s, \mathbf{y}_1(s))|| ds \leq L \int_0^t h(s) ds, \quad 0 \leq t \leq T$$

or

$$0 \leq h(t) \leq L \left| \int_0^t h(s) ds \right|, \quad |t| \leq T.$$

Since $h(0) = 0$, Grönwall's inequality implies that $h(t) = 0$ for all $|t| \leq T$. Hence $\mathbf{y}_1$ and $\mathbf{y}_2$ coincide on that interval. Since T was arbitrary, the two solutions are equal everywhere.

Exercise 3. Let $\mathbf{f}(t, \mathbf{x}) = A(t)\mathbf{x}$ where $A(t)$ is a $d \times d$ real matrix where all its components are continuous functions in t and globally bounded in t. Show that $\mathbf{f}$ verifies the conditions of Exercise 2, hence its corresponding ODE has a unique, global solution.

Exercise 4. Consider the equation

$$y'(t) = \frac{y^2(t)}{1 - y^2(t)}, \quad y(0) = 1/2.$$

1. Define $g : (-1, 1) \mapsto \mathbb{R}$, $g(x) = \frac{x^2}{1-x^2}$. Let $\mathbf{f} : \mathbb{R} \times (-1, 1) \mapsto \mathbb{R}$, $\mathbf{f}(t, x) := g(x)$. Show that $y'(t) = \mathbf{f}(t, y(t))$ and identify d, t_0, I and U.
2. Show that $\mathbf{f} \in C^1(\mathbb{R} \times U)$ and it obeys a local Lipschitz condition.
3. Show that for t near 0 we can rewrite the equation as:

$$[y(t) + 1/y(t) + t]' = 0.$$

4. Show that $y(t) + 1/y(t) = 5/2 - t$ for t near 0. Find $y(t)$ out of this algebraic equation, using that our unique solution obeys $y(0) = 1/2$.

5. Is it true that the solution can be extended to $t \in (-\infty, 1/2)$?

Exercise 5. Let $\mathbf{f} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ be a continuous function. Assume that $\mathbf{f}$ obeys a **global Lipschitz condition**, i.e. there exists a constant $C > 0$ such that

$$||\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})|| \leq C||\mathbf{x} - \mathbf{y}||, \quad \forall t \in \mathbb{R}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Consider the equation $\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t))$, where $\mathbf{y}(0) = \mathbf{y}_0$. Show that there exists a unique **global** solution $\mathbf{y} : \mathbb{R} \mapsto \mathbb{R}^d$ which solves the equation for all $t \in \mathbb{R}$.

Hint. The original differential equation is equivalent with the integral equation:

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_0^t \mathbf{f}(s, \mathbf{y}(s))ds.$$

As in Exercise 2, we assume that a solution only exists on an interval of the form $(-T_1, T_2)$ with $T := T_2 = \min\{T_1, T_2\} < \infty$. We will show that this leads to a contradiction. Define $h(t) := ||\mathbf{y}(t) - \mathbf{y}_0||$. We have:

$$\mathbf{y}(t) - \mathbf{y}_0 = \int_0^t \mathbf{f}(s, \mathbf{y}_0)ds + \int_0^t [\mathbf{f}(s, \mathbf{y}(s)) - \mathbf{f}(s, \mathbf{y}_0)]ds.$$

After taking the norms and using the Lipschitz constant (take $t > 0$):

$$h(t) \leq \int_0^t ||\mathbf{f}(s, \mathbf{y}_0)||ds + C \int_0^t h(s)ds \leq \int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds + C \int_0^t h(s)ds.$$

In general:

$$h(t) \leq \int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds + C \left| \int_0^t h(s)ds \right|, \quad |t| \leq T.$$

Up to a use of Grönwall's inequality this shows that:

$$h(t) = ||\mathbf{y}(t) - \mathbf{y}_0|| \leq \left(\int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds \right) e^{CT}, \quad |t| \leq T.$$

In other words, the solution always remains inside a closed ball with center at $\mathbf{y}_0$ and radius $R = \left(\int_0^T ||\mathbf{f}(s, \mathbf{y}_0)||ds \right) e^{CT}$. Let

$$M := \sup_{0 \leq s \leq T} \sup_{\mathbf{x} \in \overline{B_R(\mathbf{y}_0)}} ||\mathbf{f}(s, \mathbf{x})|| < \infty.$$

Choose some sequence $t_n \in (0, T)$ which converges to T. We have:

$$\mathbf{y}(t_p) - \mathbf{y}(t_q) = \int_{t_q}^{t_p} \mathbf{f}(s, \mathbf{y}(s))ds, \qquad ||\mathbf{y}(t_p) - \mathbf{y}(t_q)|| \leq M|t_p - t_q|.$$

This shows that the sequence $\{\mathbf{y}(t_n)\}_{n \geq 1}$ is Cauchy and converges to some $\mathbf{y}_T$. If $s_n \in (0, T)$ is another sequence which converges to T, we have:

$$\mathbf{y}(t_n) - \mathbf{y}(s_n) = \int_{s_n}^{t_n} \mathbf{f}(s, \mathbf{y}(s)), \qquad ||\mathbf{y}(t_n) - \mathbf{y}(s_n)|| \leq M|t_n - s_n|$$

which proves that $\mathbf{y}_T$ is independent of the sequence we choose, hence $\mathbf{y}(T - 0)$ exists and equals $\mathbf{y}_T$, and:

$$\mathbf{y}'(T - 0) = \mathbf{f}(T, \mathbf{y}_T).$$

Reasoning as in Exercise 2, we can locally extend $\mathbf{y}$ to the interval $[T, T+\delta)$, thus contradicting the maximality of the interval $(-T_1, T)$.

Now let us prove uniqueness. Assume that both $\mathbf{y}_1$ and $\mathbf{y}_2$ solve the integral equation. Define $h(t) := ||\mathbf{y}_1(t) - \mathbf{y}_2(t)||$. We have:

$$\mathbf{y}_1(t) - \mathbf{y}_2(t) = \int_0^t [\mathbf{f}(s, \mathbf{y}_1(s)) - \mathbf{f}(s, \mathbf{y}_2(s))]ds, \quad 0 \le h(t) \le C \left| \int_0^t h(s)ds \right|.$$

From Exercise 1 we get $h(t) = 0$ for all t and we are done.

Exercise 6. Let $\mathbf{g} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be given by $\mathbf{g}(\mathbf{x}) = [-x_2, x_1]$ and consider the equation

$$\mathbf{y}'(t) = \mathbf{g}(\mathbf{y}(t)), \quad \mathbf{y}(0) = [1, 0].$$

1. Write the equation on each component and show that $y_1'(t) = -y_2(t)$ and $y_2'(t) = y_1(t)$, with $y_1(0) = 1$ and $y_2(0) = 0$. Prove that $\mathbf{f}(t, \mathbf{x}) = \mathbf{g}(\mathbf{x})$ is continuous on $\mathbb{R}^3$ and obeys a global Lipschitz condition on the $\mathbf{x}$ variables. Use Exercise 2 to conclude that the solution is unique and global in time.
2. Show that $y_1^2(t) + y_2^2(t) = 1$ for all t.
3. Use the uniqueness of the equation in order to show that $y_1(t) = y_1^2(t/2) - y_2^2(t/2)$ and $y_2(t) = 2y_1(t/2)y_2(t/2)$.
4. Prove that there must exist a $T > 0$ such that $y_1(T) = 1$ and $y_2(T) = 0$.
5. Use the uniqueness of the equation in order to show that $y_1(t) = y_1(t+T)$ and $y_2(t) = y_2(t+T)$ for all t.
6. Denote by $2P$ the smallest positive T for which (4) is true. Show that $y_1(P) = -1$ and $y_2(P) = 0$.
7. Use the uniqueness of the equation and (6) to show that $y_1(t) = -y_1(P-t)$ and $y_2(t) = y_2(P-t)$.
8. Use the uniqueness of the equation and (6) to show that $y_1(t) = -y_1(P+t)$ and $y_2(t) = -y_2(P+t)$.
9. Use the uniqueness of the equation to show that $y_1(t) = y_1(-t)$ and $y_2(t) = -y_2(-t)$.
10. Put $t = P/2$ in (7) and show that $y_1(P/2) = 0$. Use this in (8) to show that $y_1(3P/2) = 0$. Use the construction in (4) to prove that $y_2(P/2) = 1$ and $y_2(3P/2) = -1$.
11. Show that $y_1'(0) = 0$, $y_1''(0) = -1$, $y_1'''(0) = 0$, $y_1^{(4)}(0) = 1$,... . Compute the Taylor series of $y_1(t)$ around 0 and show that it has an infinite radius of convergence. Can you recognize the function? What about the number P?

Hints.

(2). Show that $[y_1^2(t) + y_2^2(t)]' = 0$ for all t.

(3). Define $\tilde{y}_1(t) := y_1^2(t/2) - y_2^2(t/2)$ and $\tilde{y}_2(t) := 2y_1(t/2)y_2(t/2)$. Prove that $\tilde{\mathbf{y}}'(t) = \mathbf{g}(\tilde{\mathbf{y}}(t))$ and $\tilde{\mathbf{y}}(0) = [1, 0]$.

(4). Here it is very important to remember the identity from (2), i.e. $y_1^2(t) + y_2^2(t) = 1$. When t is slightly larger than zero, $y_2'(t) = y_1(t) \sim 1 > 0$, thus $y_2(t)$ increases from the value $y_2(0) = 0$ and becomes positive. At the same time, $y_1'(t) = -y_2(t)$ must be negative and y_1 decreases. This remains true up to some value $t_1 > 0$ where $y_1(t_1) = 0$ and $y_2(t_1) = 1$. If t is slightly larger than t_1, $y_1'(t) \sim -1$ hence y_1 continues to decrease and becomes negative. Thus $y_2'(t) = y_1(t) < 0$ and y_2 starts also to decrease. Both y_1 and y_2 decrease until t reaches some value $t_2 > t_1$ where $y_2(t_2) = 0$ and $y_1(t_2)$ must equal -1. When t is slightly larger than t_2, $y_2'(t) = y_1(t) \sim -1$ thus y_2 continues to decrease and becomes negative. Hence $y_1'(t) = -y_2(t) > 0$ which makes y_1 to increase again. There must exist a point $t_3 > t_2$ such that $y_2(t_3) = -1$ and $y_1(t) = 0$. Finally, for $t > t_3$, y_1 will continue to increase as long as y_2 is negative, up to some value $t_4 > t_3$ where $y_2(t_4) = 0$, and necessarily, $y_1(t_4) = 1$. We see that we got again the values of the initial condition and actually t_4 is the smallest positive value of t for which this happens.

(6) With the above notation, this will actually show that $2t_2 = t_4$. Replace t with $2P$ in the two identities of (3). We have $1 = y_1(2P) = y_1^2(P) - y_2^2(P) = 2y_1^2(P) - 1$ and $0 = y_2(2P) = 2y_1(P)y_2(P)$. The first identity implies that $|y_1(P)| = 1$; this implies that $y_2(P) = 0$. Hence $y_1(P)$ equals either $+1$ or -1. But it cannot equal $+1$, because we assumed that the smallest positive value of t for which we come back to the original initial condition was $2P$. Thus $y_1(P) = -1$ and $y_2(P) = 0$, which shows that P must be equal to t_2, the only point smaller than t_4 where these values are taken.