Exercises, Implicit function theorem

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Exercise 1. Let $\mathbf{h} : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $\mathbf{h}(u,v) = u^2 + (v-1)^2 - 4$. Show that $\mathbf{h}(2,1) = 0$, and $\mathbf{h} \in C^1(\mathbb{R}^2)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $f : (1-\epsilon, 1+\epsilon) \mapsto \mathbb{R}$ such that

$$\mathbf{h}(f(v), v) = 0, \quad \forall v \in (1-\epsilon, 1+\epsilon).$$

Find $f'(1)$. Can you find f explicitly in this case? Can you repeat the construction around the point $\mathbf{a} = [0,3]$?

Exercise 2. Let $\mathbf{h} : \mathbb{R}^{1+2} \mapsto \mathbb{R}$ given by $\mathbf{h}(u,\mathbf{w}) = u^2 + \mathbf{w}^2 - 4$. Show that $\mathbf{h}([2,0,0]) = 0$, and $\mathbf{h} \in C^1(\mathbb{R}^3)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $f : B_\epsilon([0,0]) \subset \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\mathbf{h}(f(\mathbf{w}), \mathbf{w}) = 0, \quad \forall \mathbf{w} \in B_\epsilon([0,0]).$$

Find $Df(\mathbf{w})$. Can you find f explicitly in this case? Can you repeat the construction around the point $\mathbf{a} = [0,2,0]$?

Exercise 3. Let $\mathbf{h} : \mathbb{R}^{2+2} \mapsto \mathbb{R}^2$ given by $\mathbf{h}(\mathbf{u},\mathbf{w}) = [u_1^2 + u_2 + w_1^2, e^{u_1} - 1 + u_2 + w_2]$. Show that $\mathbf{h}([0,0,0,0]) = [0,0]$, and $\mathbf{h} \in C^1(\mathbb{R}^4)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $\mathbf{f} : B_\epsilon([0,0]) \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$\mathbf{h}(\mathbf{f}(\mathbf{w}), \mathbf{w}) = 0, \quad \forall \mathbf{w} \in B_\epsilon([0,0]).$$

Find $D\mathbf{f}([0,0])$. Can you find $\mathbf{f}$ explicitly in this case?

Exercise 4. Assume that the parameters $\mathbf{w} = [w_1, w_2]$ belong to a neighborhood of $\mathbf{w}_0 = [0,-1]$. Consider the equations:

$$u_3^2 w_2 + u_3 w_2^2 + u_1^2 - (u_2 + w_1)^2 = -3,$$
$$e^{u_3 + w_2} - u_1 - u_2 - w_1 = -2,$$
$$(u_3 + w_2)^2 + u_1 + u_2 + w_1^2 = 3.$$

Show that if $\mathbf{w} = \mathbf{w}_0$ then by replacing $[u_1, u_2, u_3]$ with $[1,2,1]$ we get a solution for the above system. Show that if $\mathbf{w}$ is close enough to $\mathbf{w}_0$, one can still find a solution $\mathbf{u} = \mathbf{f}(\mathbf{w})$ where $\mathbf{f}$ is continuously differentiable. Find $[D\mathbf{f}(\mathbf{w}_0)]$.

Exercise 5. Under the conditions of the implicit function theorem we know that we can write:

$$[D\mathbf{f}(\mathbf{w})] = -[D_{\mathbf{u}}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])]^{-1}[D_{\mathbf{w}}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])], \quad \forall \mathbf{w} \in E.$$

Prove using the chain rule that if $\mathbf{h}$ is a C^2 function, the same is true for $\mathbf{f}$. Show that if $\mathbf{h}$ is a C^q function with $q \geq 1$, the same is true for $\mathbf{f}$.

Exercise 6. Assume that the parameters $\mathbf{w} = [w_1, w_2]$ belong to a neighborhood of $\mathbf{w}_0 = [1,1]$. Consider the equations:

$$w_1^2 + w_2^2 + u_1^2 + u_2^2 = 3,$$
$$w_1 + w_2 + u_1 + u_2 = 3.$$

Show that if $\mathbf{w} = \mathbf{w}_0$, then by replacing $[u_1, u_2]$ with $[0,1]$, the system is solved. Show that if $\mathbf{w}$ is close enough to $\mathbf{w}_0$, one can still find a solution $\mathbf{u} = \mathbf{f}(\mathbf{w})$ where $\mathbf{f}$ is continuously differentiable. Prove that $\mathbf{f}$ is C^2. Find all first and second order partial derivatives of $\mathbf{f}$ at $\mathbf{w}_0$.