Exercises, Implicit function theorem

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Exercise 1. Let $\mathbf{h} : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $\mathbf{h}(u, v) = u^2 + (v-1)^2 - 4$. Show that $\mathbf{h}(2, 1) = 0$, and $\mathbf{h} \in C^1(\mathbb{R}^2)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $f : (1-\epsilon, 1+\epsilon) \mapsto \mathbb{R}$ such that

$$\mathbf{h}(f(v), v) = 0, \quad \forall v \in (1-\epsilon, 1+\epsilon).$$

Find $f'(1)$. Can you find f explicitly in this case? Can you repeat the construction around the point $\mathbf{a} = [0, 3]$?

Exercise 2. Let $\mathbf{h} : \mathbb{R}^{1+2} \mapsto \mathbb{R}$ given by $\mathbf{h}(u, \mathbf{w}) = u^2 + \mathbf{w}^2 - 4$. Show that $\mathbf{h}([2, 0, 0]) = 0$, and $\mathbf{h} \in C^1(\mathbb{R}^3)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $f : B_\epsilon([0, 0]) \subset \mathbb{R}^2 \mapsto \mathbb{R}$ such that

$$\mathbf{h}(f(\mathbf{w}), \mathbf{w}) = 0, \quad \forall \mathbf{w} \in B_\epsilon([0, 0]).$$

Find $Df(\mathbf{w})$. Can you find f explicitly in this case? Can you repeat the construction around the point $\mathbf{a} = [0, 2, 0]$?

Exercise 3. Let $\mathbf{h} : \mathbb{R}^{2+2} \mapsto \mathbb{R}^2$ given by $\mathbf{h}(\mathbf{u}, \mathbf{w}) = [u_1^2 + u_2 + w_1^2, e^{u_1} - 1 + u_2 + w_2]$. Show that $\mathbf{h}([0, 0, 0, 0]) = [0, 0]$, and $\mathbf{h} \in C^1(\mathbb{R}^4)$. Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $\mathbf{f} : B_\epsilon([0, 0]) \subset \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that

$$\mathbf{h}(\mathbf{f}(\mathbf{w}), \mathbf{w}) = 0, \quad \forall \mathbf{w} \in B_\epsilon([0, 0]).$$

Find $D\mathbf{f}([0, 0])$. Can you find $\mathbf{f}$ explicitly in this case?