Exercises: Pointwise and Uniform convergence

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Let $\{f_n\}_{n\geq 1}$ be a sequence of real functions such that $f_n : I \mapsto \mathbb{R}$. We say that this sequence of functions converges pointwise on I if the sequence of real numbers $\{f_n(x)\}_{n\geq 1} \subset \mathbb{R}$ is convergent for every $x \in I$. The pointwise limit defines a function as follows:

$$P(x) := \lim_{n\to\infty} f_n(x), \quad \forall x \in I.$$

We say that the sequence $\{f_n\}_{n\geq 1}$ converges uniformly on I towards a function $U : I \mapsto \mathbb{R}$ if the real sequence $\{a_n\}_{n\geq 1}$ defined by

$$a_n := \sup_{t\in I} |f_n(t) - U(t)|, \quad n \geq 1$$

is convergent to zero. In other words, given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\sup_{t\in I} |f_n(t) - U(t)| < \epsilon \quad \text{whenever} \quad n \geq N_\epsilon. \tag{0.1}$$

Exercise 1. Let $f_n : [0,1] \mapsto \mathbb{R}$, $n \geq 1$, with $f_n(x) = \frac{nx}{nx+1}$. Show that the sequence has a pointwise limit and compute it.

Hint: If $x = 0$ we have $f_n(0) = 0$ for all n. Thus $\{f_n(0)\}_{n\geq 1}$ is convergent and its limit is $P(0) = 0$. If $0 < x \leq 1$ we have:

$$f_n(x) = \frac{x}{x + 1/n}, \quad \lim_{n\to\infty} f_n(x) = P(x) = 1.$$

Exercise 2. Show that if a sequence of functions has a uniform limit U, then the sequence is also pointwise convergent and $P = U$.

Hint: Fix an arbitrary $x \in I$. Let $\epsilon > 0$. From (0.1) it follows that there exists some N_ϵ such that

$$|f_n(x) - U(x)| \leq \sup_{t\in I} |f_n(t) - U(t)| < \epsilon \quad \text{whenever} \quad n \geq N_\epsilon.$$

This shows that $\{f_n(x)\}_{n\geq 1}$ converges to $U(x)$ for every x.

Exercise 3. Consider $f_n : [0,1] \mapsto \mathbb{R}$, $n \geq 1$, with $f_n(x) = \frac{nx}{nx+1}$. Does it have a uniform limit?

Hint: From Exercise 1 we know that the sequence has a pointwise limit P which is given by $P(0) = 0$ and $P(x) = 1$ if $0 < x \leq 1$. From Exercise 2 we know that IF the sequence has some uniform limit U, then it must be equal with P. Let us show that the sequence does NOT converge uniformly to P.

If $n \geq 1$ we have that $0 < 1/n \leq 1$ and $f_n(1/n) = 1/2$ and $P(1/n) = 1$. Hence:

$$1/2 = |f_n(1/n) - P(1/n)| \leq \sup_{t\in[0,1]} |f_n(t) - P(t)| =: a_n.$$

In other words, the sequence a_n cannot converge to zero, hence P is not a uniform limit.

Exercise 4. Consider $f_n : [0,1] \mapsto \mathbb{R}$, $n \geq 1$, with $f_n(x) = \frac{nx}{n+x}$. Does it have a uniform limit?

Hint: We have

$$f_n(x) = \frac{x}{1 + x/n}, \quad \lim_{n\to\infty} f_n(x) = P(x) = x, \quad \forall x \in [0,1].$$

Moreover:

$$|f_n(t) - P(t)| = \frac{t^2}{n+t} \leq \frac{1}{n}, \quad \forall t \in [0,1]$$

hence

$$a_n = \sup_{t\in[0,1]} |f_n(t) - P(t)| \leq \frac{1}{n}$$

converges to zero.

Exercise 5. Consider $f_n :]0,1[\mapsto \mathbb{R}$, $n \geq 1$, with $f_n(x) = \frac{nx}{nx^2+1}$. Does it have a uniform limit?

Exercise 6. Consider the metric space $X = C([0,1];\mathbb{R})$ consisting of continuous real functions $f : [0,1] \mapsto \mathbb{R}$, where the distance between two elements f and g of X is given by:

$$d(f,g) := \sup_{t\in[0,1]} |f(t) - g(t)|.$$

Denote by $\mathbf{0}$ the zero function, i.e. $\mathbf{0}(x) = 0$ for all $x \in [0,1]$.

(i). Show that the set $A := \{f \in X : |f(x)| \leq 1, \forall x \in [0,1]\}$ is bounded and closed.

(ii). Consider the sequence $\{f_n\}_{n\geq 1} \subset X$ where

$$f_n(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq 2^{-n-1}; \\ 2^{n+2}x - 2 & \text{if } 2^{-n-1} < x \leq \frac{3}{2^{n+2}}; \\ -2^{n+2}x + 4 & \text{if } \frac{3}{2^{n+2}} < x < 2^{-n}; \\ 0 & \text{if } 2^{-n} \leq x \leq 1. \end{cases}$$

Draw the graphs of f_1 and f_2. Show that each $f_n \in A$. Show that the pointwise limit of this sequence is the $\mathbf{0}$ function. Is the sequence uniform convergent?

(iii). Show that $d(f_n, f_m) = 1$ for all $n \neq m$. Use this in order to conclude that no subsequence $\{f_{n_k}\}_{k\geq 1}$ can have the Cauchy property.

(iv). Show that one cannot find a subsequence $\{f_{n_k}\}_{k\geq 1}$ which converges to an element of A. Conclude that A is not compact. Does this contradict the Heine-Borel theorem?

Hint (i). For boundedness, show that $A \subset B_2(\mathbf{0}) := \{f \in X : d(f,\mathbf{0}) < 2\}$. In order to show that A is closed, let $g \in \overline{A}$. There exists a sequence $\{g_n\}_{n\geq 1} \subset A$ such that $\lim_{n\to\infty} d(g_n, g) = 0$. In particular, this means that $\lim_{n\to\infty} |g_n(t) - g(t)| = 0$ for all t and g is continuous. Moreover, from the inequality:

$$|g(t)| \leq |g_n(t) - g(t)| + |g_n(t)| \leq |g_n(t) - g(t)| + 1$$

by taking n to infinity we conclude that $|g(t)| \leq 1$ for all t, thus $g \in A$.

Exercise 7. This exercise is NOT trivial, especially in its second part. It is about the construction of the simplest infinitely dimensional, real and separable Hilbert space (a vector space with an inner product, whose associated normed space is complete, and which has a countable orthonormal basis). This space plays a fundamental role in quantum mechanics, signal processing, probability theory, functional analysis, operator theory. If you find that the proof of (vi) is nice and natural, you should definitely consider writing a master thesis in analysis.

Consider the set

$$l^2(\mathbb{N}) := \left\{ \{x(n)\}_{n\geq 1} : x(n) \in \mathbb{R} \quad \forall n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty} [x(n)]^2 < \infty \right\}.$$

(i). Let $\alpha, \beta \in \mathbb{R}$. Prove that if $\mathbf{x} := \{x(n)\}_{n\geq 1} \in l^2(\mathbb{N})$ and $\mathbf{y} := \{y(n)\}_{n\geq 1} \in l^2(\mathbb{N})$, then $\mathbf{z} := \alpha\mathbf{x} + \beta\mathbf{y}$ with $z(n) := \alpha x(n) + \beta y(n)$ also belongs to $l^2(\mathbb{N})$.

(ii). Prove that if $\mathbf{x}, \mathbf{y} \in l^2(\mathbb{N})$ then the series $\sum_{n=1}^{\infty} x(n)y(n)$ is absolutely convergent.
(iii). Prove that $\langle \mathbf{x}|\mathbf{y}\rangle := \sum_{n=1}^{\infty} x(n)y(n)$ defines a scalar (inner) product on $l^2(\mathbb{N})$. Conclude that $l^2(\mathbb{N})$ is a normed vector space with the norm $||\mathbf{x}|| := \sqrt{\langle \mathbf{x}|\mathbf{x}\rangle} = \sqrt{\sum_{n=1}^{\infty}[x(n)]^2}$. Does it have a finite dimension?
(iv). Show that $l^2(\mathbb{N})$ is a Banach space.
(v). Consider the set

$$A := \left\{\mathbf{x} \in l^2(\mathbb{N}) : \ ||\mathbf{x}|| \leq 1\right\}.$$

Show that A is bounded and closed, but not compact.
(vi). Consider the set

$$B := \left\{\mathbf{x} \in l^2(\mathbb{N}) : \ |x(j)| \leq 1/j, \quad \forall j \geq 1\right\}.$$

Show that B is compact. Is B bounded and closed?

Hint (i). Derive the inequality $[z(n)]^2 \leq 2\alpha^2[x(n)]^2 + 2\beta^2[y(n)]^2$.

Hint (ii). Let $s_N := \sum_{m=1}^{N} |x(m)|\ |y(m)|$ for all $N \geq 1$. Use the Cauchy-Schwarz inequality in $\mathbb{R}^N$ in order to show that

$$0 \leq s_N \leq \sqrt{\sum_{n=1}^{N}[x(n)]^2}\ \sqrt{\sum_{n=1}^{N}[y(n)]^2} \leq \sqrt{\sum_{n=1}^{\infty}[x(n)]^2}\ \sqrt{\sum_{n=1}^{\infty}[y(n)]^2}$$

for all N.

Hint (iv). We have to show that every Cauchy sequence in $l^2(\mathbb{N})$ is convergent. If the sequence $\{\mathbf{x}_n\}_{n\geq 1} \subset l^2(\mathbb{N})$ is Cauchy, then given $\epsilon > 0$ there exists $N(\epsilon) \geq 1$ such that

$$||\mathbf{x}_p - \mathbf{x}_q|| < \epsilon \quad \text{whenever} \quad p > q \geq N(\epsilon). \tag{0.2}$$

We also know that every Cauchy sequence is bounded, i.e. there exists some $M < \infty$ such that

$$||\mathbf{x}_n|| \leq M, \quad \forall n \geq 1. \tag{0.3}$$

If $j \in \mathbb{N}$ is fixed, consider the real sequence $\{x_n(j)\}_{n\geq 1} \subset \mathbb{R}$. From (0.2) we have:

$$|x_p(j) - x_q(j)|^2 \leq ||\mathbf{x}_p - \mathbf{x}_q||^2 < \epsilon^2 \quad \text{whenever} \quad p > q \geq N(\epsilon).$$

Thus $\{x_n(j)\}_{n\geq 1} \subset \mathbb{R}$ is Cauchy in $\mathbb{R}$, hence it converges to some limit which we denote by $x(j) = \lim_{n\to\infty} x_n(j)$. Let us show that $\mathbf{x} := \{x(j)\}_{j\geq 1}$ belongs to $l^2(\mathbb{N})$. Fix some $K \geq 1$. We have:

$$\sum_{j=1}^{K}[x(j)]^2 \leq 2\sum_{j=1}^{K}[x(j) - x_n(j)]^2 + 2\sum_{j=1}^{K}[x_n(j)]^2 \leq 2\sum_{j=1}^{K}[x(j) - x_n(j)]^2 + 2||\mathbf{x}_n||^2$$

or using (0.3) we obtain:

$$\sum_{j=1}^{K}[x(j)]^2 \leq 2\sum_{j=1}^{K}[x(j) - x_n(j)]^2 + 2M^2, \quad \forall n \geq 1.$$

Taking n to infinity on the right hand side gives:

$$\sum_{j=1}^{K}[x(j)]^2 \leq 2M^2.$$

Since the estimate is independent of K, the series giving $||\mathbf{x}||$ converges. Next we prove that $\mathbf{x}$ is the limit of $\{\mathbf{x}_n\}_{n\geq 1} \subset l^2(\mathbb{N})$. For every $j \geq 1$ we have:

$$[x(j) - x_n(j)]^2 \leq 2[x(j) - x_p(j)]^2 + 2[x_p(j) - x_n(j)]^2,$$

which leads to (here $K \geq 1$ is arbitrary but fixed):

$$\sum_{j=1}^{K}[x(j) - x_n(j)]^2 \leq 2\sum_{j=1}^{K}[x(j) - x_p(j)]^2 + 2||\mathbf{x}_p - \mathbf{x}_n||^2, \quad \forall n, p \geq 1.$$

Given $\epsilon > 0$, choose both n and p larger than $N(\epsilon/2)$ as given by (0.2). Then:

$$\sum_{j=1}^{K}[x(j) - x_n(j)]^2 \leq 2\sum_{j=1}^{K}[x(j) - x_p(j)]^2 + \epsilon^2/2, \quad \forall n, p \geq N(\epsilon/2).$$

Now let p tend to infinity on the right hand side. We obtain:

$$\sum_{j=1}^{K}[x(j) - x_n(j)]^2 \leq \epsilon^2/2, \quad \forall n \geq N(\epsilon/2),$$

where the estimate is independent of K. We conclude that $||\mathbf{x} - \mathbf{x}_n|| \leq \epsilon/\sqrt{2}$ for all $n \geq N(\epsilon/2)$ and we are done.

Hint (v). Show that $A \subset B_2(\mathbf{0})$ (which is the open ball with radius 2 and centered at the origin $\mathbf{0} \in l^2(\mathbb{N})$; here $\mathbf{0}$ is the element whose elements are all equal to zero). Thus A is bounded. To show that A is closed, let $\mathbf{x} \in \overline{A}$ and consider a sequence $\{\mathbf{x}_n\}_{n\geq 1} \subset A$ which converges to $\mathbf{x}$, i.e. $\lim_{n\to\infty} ||\mathbf{x}_n - \mathbf{x}|| = 0$. We have

$$||\mathbf{x}|| \leq ||\mathbf{x} - \mathbf{x}_n|| + ||\mathbf{x}_n|| \leq ||\mathbf{x} - \mathbf{x}_n|| + 1, \quad \forall n \geq 1.$$

Conclude from here that $\mathbf{x} \in A$.

Now let us show that A is not sequentially compact. Consider the sequence $\{\delta_n\}_{n\geq 1} \subset A$ where the elements δ_n have the components $\delta_n(j) = 0$ if $j \neq n$ and $\delta_n(n) = 1$. Show that $\langle \delta_m | \delta_n \rangle = 0$ and $||\delta_n - \delta_m|| = \sqrt{2}$ for every $m \neq n$. Can such a sequence have a convergent subsequence?

Hint (vi). We show that B is sequentially compact, i.e. given an arbitrary sequence $\{\mathbf{x}_n\}_{n\geq 1} \subset B$ we will construct a subsequence $\{\mathbf{x}_{N_k}\}_{k\geq 1}$ and an $\mathbf{x} \in B$ such that $\lim_{k\to\infty} ||\mathbf{x}_{N_k} - \mathbf{x}|| = 0$. The argument is rather involved but also standard in analysis (one of the 10 standard tricks which a specialist must know), and it is based on a 'diagonal construction'.

From the definition of B we know that $|x_n(1)| \leq 1$ for all $n \geq 1$. Using the Bolzano-Weierstrass theorem in $\mathbb{R}$ we can find a subsequence $\{x_{n_a}(1)\}_{a\geq 1} \subset \{x_n(1)\}_{n\geq 1}$ which converges to some real number (denoted by $x(1) \in [-1, 1]$), i.e. $\lim_{a\to\infty} |x_{n_a}(1) - x(1)| = 0$. Define $N_1 := n_1$.

Again from the definition of B, we know that $|x_{n_a}(2)| \leq 1/2$ for all $a \geq 1$. Using the Bolzano-Weierstrass theorem in $\mathbb{R}$ we can find a subsequence $\{x_{n_{a_b}}(2)\}_{b\geq 1} \subset \{x_{n_a}(2)\}_{a\geq 1}$ which converges to a point $x(2) \in [-1/2, 1/2]$, i.e. $\lim_{b\to\infty} |x_{n_{a_b}}(2) - x(2)| = 0$. It is very important to notice that the subsequence $\{x_{n_{a_b}}(1)\}_{b\geq 1} \subset \{x_{n_a}(1)\}_{a\geq 1}$ also converges to $x(1)$. Hence we can write:

$$\lim_{b\to\infty} |x_{n_{a_b}}(1) - x(1)| = 0 \quad \text{and} \quad \lim_{b\to\infty} |x_{n_{a_b}}(2) - x(2)| = 0.$$

Thus there exists some B sufficiently large such that the following three inequalities take place simultaneously:

$$N_1 < n_{a_B}, \quad |x_{n_{a_B}}(1) - x(1)| \leq 1/2, \quad |x_{n_{a_B}}(2) - x(2)| \leq 1/2.$$

Now we can define $N_2 := n_{a_B}$. Using the same strategy, i.e. considering the sequence $\{x_{n_{a_b}}(3)\}_{b\geq 1}$ we can construct $x(3)$ as a limit of some subsequence $\{x_{n_{a_{b_c}}}(3)\}_{c\geq 1}$, and:

$$\lim_{c\to\infty} |x_{n_{a_{b_c}}}(1) - x(1)|, \quad \lim_{c\to\infty} |x_{n_{a_{b_c}}}(2) - x(2)|, \quad \lim_{c\to\infty} |x_{n_{a_{b_c}}}(3) - x(3)| = 0.$$

Thus we may find some C large enough such that:

$$N_2 < n_{a_{b_C}}, \ |x_{n_{a_{b_c}}}(1) - x(1)| \leq 1/3, \ |x_{n_{a_{b_C}}}(2) - x(2)| \leq 1/3, \ |x_{n_{a_{b_C}}}(3) - x(3)| \leq 1/3.$$

We put $N_3 := n_{a_{b_C}}$. This construction can be repeated indefinitely for every $k \geq 1$ and define $x(k) \in [-1/k, 1/k]$ to be the limit of some subsequence $\{x_{n_{a_{b_{\dots z}}}}(k)\}_{z\geq 1}$. If $j \leq k$, then $x_{n_{a_{b_{\dots z}}}}(j)$ will also converge to $x(j)$. Thus given $k \geq 4$ we can find some N_k such that:

$$N_{k-1} < N_k, \quad |x_{N_k}(j) - x(j)| \leq 1/k, \quad 1 \leq j \leq k. \tag{0.4}$$

Now we will prove that $\mathbf{x} = \{x(j)\}_{j\geq 1}$ is the limit of $\{\mathbf{x}_{N_k}\}_{k\geq 1}$. Fix some $\epsilon > 0$. Because the series $\sum_{j\geq 1} 1/j^2$ is convergent, it follows that

$$\lim_{J\to\infty} \sum_{j>J}^{\infty} 1/j^2 = 0.$$

Thus there exists some J_ϵ large enough such that

$$0 < \sum_{j>J_\epsilon}^{\infty} 1/j^2 < \epsilon^2/8.$$

We have:

$$\begin{aligned}
||\mathbf{x}_{N_k} - \mathbf{x}||^2 &= \sum_{j=1}^{J_\epsilon} |x_{N_k}(j) - x(j)|^2 + \sum_{j>J_\epsilon} |x_{N_k}(j) - x(j)|^2 \\
&\leq \sum_{j=1}^{J_\epsilon} |x_{N_k}(j) - x(j)|^2 + 2 \sum_{j>J_\epsilon} (|x_{N_k}(j)|^2 + |x(j)|^2) \\
&\leq \sum_{j=1}^{J_\epsilon} |x_{N_k}(j) - x(j)|^2 + 4 \sum_{j>J_\epsilon} 1/j^2 \leq \sum_{j=1}^{J_\epsilon} |x_{N_k}(j) - x(j)|^2 + \epsilon^2/2.
\end{aligned}$$

Thus:

$$||\mathbf{x}_{N_k} - \mathbf{x}||^2 \leq \sum_{j=1}^{J_\epsilon} |x_{N_k}(j) - x(j)|^2 + \epsilon^2/2, \quad \forall k \geq 1.$$

The above inequality holds true for all $k \geq 1$. If we demand that $k \geq J_\epsilon$, using (0.4) in the above inequality gives:

$$||\mathbf{x}_{N_k} - \mathbf{x}||^2 \leq \frac{J_\epsilon}{k^2} + \epsilon^2/2, \quad \forall k \geq J_\epsilon.$$

Now we can find some $K_\epsilon \geq J_\epsilon$ large enough such that $\frac{J_\epsilon}{k^2} < \epsilon^2/2$ whenever $k \geq K_\epsilon$, which leads to

$$||\mathbf{x}_{N_k} - \mathbf{x}|| < \epsilon \quad \text{whenever} \quad k \geq K_\epsilon$$

and we are done.