Power series are analytic

Horia Cornean[1]

1 The exponential and the logarithm

For every $x \in \mathbb{R}$ we define the function given by

$$\exp(x) := 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n \geq 0} \frac{x^n}{n!}.$$

If $x = 0$ we have $\exp(0) = 1$. If $x \neq 0$, consider the series $\sum_{n\geq 0} \alpha_n$ given by $\alpha_n = \frac{x^n}{n!}$. Using the ratio criterion we have:

$$\frac{|\alpha_{n+1}|}{|\alpha_n|} = \frac{|x|}{n+1} \to 0 < 1 \quad \text{when} \quad n \to \infty,$$

which shows that the series defining $\exp(x)$ converges absolutely for all $x \in \mathbb{R}$.

We want to prove that the exponential is everywhere differentiable. Fix $a \in \mathbb{R}$ and let $h \in \mathbb{R}$. Define the function

$$F(h) := (h + a)^n, \quad n \geq 2. \tag{1.1}$$

The Taylor formula with remainder provides us with a $c = c_{n,a,h}$ between 0 and h such that $F(h) = F(0) + F'(0)h + F''(c)h^2/2$, or:

$$(h + a)^n - a^n = na^{n-1}h + \frac{n(n-1)}{2} h^2 (c_{n,a,h} + a)^{n-2}, \tag{1.2}$$

which leads to:

$$\frac{(h + a)^n}{n!} - \frac{a^n}{n!} = \frac{a^{n-1}}{(n-1)!} h + \frac{(c_{n,a,h} + a)^{n-2}}{2(n-2)!} h^2, \quad n \geq 2. \tag{1.3}$$

Thus if $h \neq 0$:

$$\frac{\exp(h + a) - \exp(a)}{h} = 1 + \frac{1}{h} \sum_{n\geq 2} \left(\frac{(h + a)^n}{n!} - \frac{a^n}{n!} \right) = 1 + \sum_{n\geq 2} \frac{a^{n-1}}{(n-1)!} + \frac{h}{2} \sum_{n\geq 2} \frac{(c_{n,a,h} + a)^{n-2}}{(n-2)!}.$$

Note first that $1 + \sum_{n\geq 2} \frac{a^{n-1}}{(n-1)!} = \exp(a)$. Moreover, since $|c_{n,a,h} + a| \leq |a| + |h|$ we may write:

$$\left| \frac{\exp(h + a) - \exp(a)}{h} - \exp(a) \right| \leq \frac{|h|}{2} \sum_{n\geq 2} \frac{(|a| + |h|)^{n-2}}{(n-2)!} = \frac{|h| \exp(|a| + |h|)}{2} \leq \frac{|h| \exp(|a| + 1)}{2},$$

which holds for every $0 < |h| \leq 1$. It follows that the exponential function is differentiable at a and $\exp'(a) = \exp(a)$.

Theorem 1.1. *We have that* $\exp(-x)\exp(x) = 1$ *and* $\exp(x) > 0$ *for all* $x \in \mathbb{R}$*. Moreover,* $\exp(a + b) = \exp(a)\exp(b)$ *for all* $a, b \in \mathbb{R}$*. Define the logarithm function*

$$\ln(x) := \int_1^x \frac{1}{t} dt, \quad x > 0.$$

Then we have $\ln(\exp(x)) = x$ *for all* $x \in \mathbb{R}$*, and* $\exp(\ln(x)) = x$ *for all* $x > 0$.

[1]IMF, AAU, February 18, 2015

Proof. We know that $\exp(0) = 1$ and $\exp'(x) = \exp(x)$ holds on $\mathbb{R}$. Define the function $f(x) = \exp(-x)\exp(x)$. Then f is differentiable and

$$f(0) = 1, \quad f'(x) = 0, \quad \forall x \in \mathbb{R}.$$

Hence $f(x) = 1$ on $\mathbb{R}$, which proves that $\exp(-x)\exp(x) = 1$ for all $x \in \mathbb{R}$. The same identity shows that $\exp(x)$ can never be zero. Now since $\exp(0) = 1 > 0$ and because exp is continuous (being differentiable), it cannot change sign because it would have to go through a zero (remember the intermediate value theorem). Hence $\exp(x) > 0$ on $\mathbb{R}$.

Now define the function $g(x) = \exp(-x - b)\exp(x)\exp(b)$ for some fixed b. We again have $g(0) = 1$ and $g'(x) = 0$ for all $x \in \mathbb{R}$, hence $\exp(-x - b)\exp(x)\exp(b) = 1$ on $\mathbb{R}$. Multiply with $\exp(x + b)$ on both sides and obtain $\exp(x)\exp(b) = \exp(x + b)$ on $\mathbb{R}$.

The logarithm function is defined to be a primitive of $1/x$, i.e.:

$$\ln'(x) = \frac{1}{x}, \quad \ln(1) = 0.$$

Define $f(x) = \ln(\exp(x)) - x$ on $\mathbb{R}$, which is possible because $\exp(x) > 0$. We have $f(0) = 0$ and $f'(x) = 0$ for all $x \in \mathbb{R}$, hence $\ln(\exp(x)) = x$ on $\mathbb{R}$.

If $x > 0$, consider the function $f(x) = \frac{1}{x}\exp(\ln(x))$. We have that $f(1) = 1$ and $f'(x) = 0$ for all $x > 0$, hence $\exp(\ln(x)) = x$ for all $x > 0$.

We have just proved that the exponential and the logarithm are inverses to each other. □

Corollary 1.2. *We have* $\ln(ab) = \ln(a) + \ln(b)$ *for all* $a, b > 0$. *Moreover,* $\ln(y^x) = x\ln(y)$ *for all* $y > 0$ *and* $x \in \mathbb{R}$. *Thus if* $y > 0$ *and* $x \in \mathbb{R}$, *we have* $y^x = \exp(x\ln(y))$.

Proof. Since

$$\exp(\ln(ab)) = ab = \exp(\ln(a))\exp(\ln(b)) = \exp(\ln(a) + \ln(b)),$$

we must have $\ln(ab) = \ln(a) + \ln(b)$ due to the injectivity of exp. If $ab = 1$ we have $0 = \ln(a) + \ln(a^{-1})$, or $\ln(a^{-1}) = -\ln(a)$. Now if $a = b$ we get $\ln(a^2) = 2\ln(a)$. By induction, we obtain that $\ln(a^n) = n\ln(a)$ for all $n \in \mathbb{N}$. Replacing a in the last identity with $b^{1/n}$ we obtain $\ln(b^{1/n}) = \frac{1}{n}\ln(b)$. Thus $\ln(b^{\frac{m}{n}}) = \frac{m}{n}\ln(b)$. Moreover, $\ln(b^{-\frac{m}{n}}) = -\frac{m}{n}\ln(b)$.

Thus we have just proved that for every rational number r and for every positive number $y > 0$ we have $\ln(y^r) = r\ln(y)$. This implies $y^r = \exp(r\ln(y))$ for every rational number r. Finally, we use that every real number x is the limit of a sequence of rational numbers, together with the continuity of exp. □

Corollary 1.3. *Let* $\alpha, \beta, \gamma > 0$. *We have that*

$$\lim_{x\to\infty} \frac{x^\alpha}{\exp(\beta x)} = \lim_{x\to\infty} \frac{\ln(x)}{x^\gamma} = 0. \tag{1.4}$$

Proof. Let N be an integer such that $\alpha < N$. We have the inequality:

$$\exp(\beta x) \geq 1 + \beta x + \cdots + \frac{\beta^N x^N}{N!} \geq \frac{\beta^N x^N}{N!}, \quad \forall x > 0.$$

Then:

$$0 \leq \frac{x^\alpha}{\exp(\beta x)} \leq \frac{N!}{\beta^N x^{N-\alpha}} \to 0 \quad \text{when} \quad x \to \infty.$$

Now if $\gamma > 0$ and $x > 0$ we have $x^\gamma = \exp(\gamma\ln(x))$. Denote by $y = \ln(x)$. Then we have:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^\gamma} = \lim_{y\to\infty} \frac{y}{\exp(\gamma y)} = 0.$$

□

2 The binomial identity

Theorem 2.1. *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then:*

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}.$$

Proof. Let $P : \mathbb{R} \mapsto \mathbb{R}$ given by $P(x) = (x+b)^n$. We have that $P'(x) = n(x+b)^{n-1}$, $P''(x) = n(n-1)(x+b)^{n-2}$, and by induction we can prove:

$$P^{(k)}(x) = n(n-1)\dots(n-k+1)(x+b)^{n-k} = \frac{n!}{(n-k)!}(x+b)^{n-k}, \quad 0 \le k \le n.$$

Moreover, $P^{(k)}(x) = 0$ if $k > n$. The Taylor formula with remainder provides us with some c between 0 and x such that:

$$P(x) = P(0) + \sum_{k=1}^{n} \frac{P^{(k)}(0)}{k!} x^k + \frac{P^{(n+1)}(c)}{(n+1)!} x^k = P(0) + \sum_{k=1}^{n} \frac{P^{(k)}(0)}{k!} x^k.$$

The final result is obtained by replacing x with a. $\square$

3 Fubini's theorem for double series

Theorem 3.1. *Let $\{\alpha_{nm}\}_{n,m\ge 0}$ be a real sequence indexed by two indices. Assume that the series $\sum_{m\ge 0} |\alpha_{nm}|$ is convergent for all n and*

$$C := \sum_{n\ge 0} \Big(\sum_{m\ge 0} |\alpha_{nm}| \Big) < \infty. \tag{3.5}$$

Then we have that $\sum_{n\ge 0} |\alpha_{nm}|$ converges for all m and:

$$\sum_{m\ge 0} \Big(\sum_{n\ge 0} |\alpha_{nm}| \Big) = C. \tag{3.6}$$

Moreover,

$$\lim_{N\to\infty} \sum_{m\ge 0} \Big(\sum_{n>N} |\alpha_{nm}| \Big) = \lim_{M\to\infty} \sum_{n\ge 0} \Big(\sum_{m>M} |\alpha_{nm}| \Big) = 0. \tag{3.7}$$

Finally,

$$\sum_{m\ge 0} \Big(\sum_{n\ge 0} \alpha_{nm} \Big) = \sum_{n\ge 0} \Big(\sum_{m\ge 0} \alpha_{nm} \Big) \in \mathbb{R}. \tag{3.8}$$

Proof. We recall a few fundamental results. If $a_n \ge 0$ is a nonnegative sequence, we define $s_N = \sum_{n=0}^{N} a_n$ to be an increasing sequence of partial sums. Then $\sum_{n\ge 0} a_n = \lim_{N\to\infty} s_N$ exists and is finite if and only if the sequence $\{s_N\}_{N\ge 0}$ is bounded from above. Moreover, if s_N converges then it is Cauchy, hence for all $\epsilon > 0$ there exists $N_\epsilon \ge 0$ such that $0 \le s_{N+k} - s_N < \epsilon$ for all $k \ge 1$ and $N \ge N_\epsilon$. This implies:

$$0 \le s_{N+k} - s_N = \sum_{n=N+1}^{N+k} a_n < \epsilon, \quad \forall k \ge 1.$$

Taking the supremum over k we get $0 \le \sum_{n\ge N+1} a_n \le \epsilon$ for every $N \ge N_\epsilon$. In other words:

$$\lim_{N\to\infty} \sum_{n>N} a_n = 0. \tag{3.9}$$

If N and M are finite natural numbers, then we have:

$$\sum_{m=0}^{M}\sum_{n=0}^{N}|\alpha_{nm}| = \sum_{n=0}^{N}\sum_{m=0}^{M}|\alpha_{nm}| \le \sum_{n=0}^{N}\sum_{m\ge 0}|\alpha_{nm}| \le C. \tag{3.10}$$

In the last two inequalities we employed the assumption (3.5). Hence

$$\sum_{m=0}^{M}\sum_{n=0}^{N}|\alpha_{nm}| \le C < \infty, \quad \forall N, M \ge 0. \tag{3.11}$$

In particular,

$$\sum_{n=0}^{N}|\alpha_{nm}| \le C < \infty, \quad \forall N, m \ge 0.$$

This shows that $\sum_{n\ge 0}|\alpha_{nm}|$ is convergent for all $m \ge 0$. Now we can take the limit $N \to \infty$ in (3.11) and obtain:

$$\sum_{m=0}^{M}\sum_{n\ge 0}|\alpha_{nm}| \le C < \infty, \quad \forall M \ge 0.$$

But this shows that the sequence of the partial sums generated by $a_m := \sum_{n\ge 0}|\alpha_{nm}|$ is bounded, hence

$$D := \sum_{m\ge 0}\big(\sum_{n\ge 0}|\alpha_{nm}|\big) \le C.$$

Now using again the first identity in (3.10) we have:

$$\sum_{n=0}^{N}\sum_{m=0}^{M}|\alpha_{nm}| = \sum_{m=0}^{M}\sum_{n=0}^{N}|\alpha_{nm}| \le \sum_{m=0}^{M}\sum_{n\ge 0}|\alpha_{nm}| \le D$$

or

$$\sum_{n=0}^{N}\sum_{m=0}^{M}|\alpha_{nm}| \le D, \quad \forall N, M \ge 0.$$

Our hypothesis guarantees that $\lim_{M\to\infty}\sum_{m=0}^{M}|\alpha_{nm}|$ exists and is finite, hence:

$$\sum_{n=0}^{N}\sum_{m\ge 0}|\alpha_{nm}| \le D, \quad \forall N \ge 0.$$

Thus by taking $N \to \infty$ we get:

$$C = \sum_{n\ge 0}\sum_{m\ge 0}|\alpha_{nm}| \le D$$

which proves that $C = D$.

Now we have to prove (3.7). Define $\beta_{nm} = \alpha_{nm}$ if $n > N$, and $\beta_{nm} = 0$ if $0 \le n \le N$. Then we have:

$$\sum_{m\ge 0}\sum_{n\ge 0}|\beta_{nm}| = \sum_{n\ge 0}\sum_{m\ge 0}|\beta_{nm}| \quad \text{or} \quad \sum_{m\ge 0}\sum_{n>N}|\alpha_{nm}| = \sum_{n>N}\big(\sum_{m\ge 0}|\alpha_{nm}|\big).$$

Denoting by $a_n = \sum_{m\ge 0}|\alpha_{nm}|$ we see that (use (3.9)):

$$\sum_{m\ge 0}\sum_{n>N}|\alpha_{nm}| = \sum_{n>N}a_n \to 0 \quad \text{when} \quad N \to \infty.$$

In a similar way we can prove the other limit in (3.7).

Now we have to prove (3.8). First of all, because

$$\Big|\sum_{m\geq 0}\alpha_{nm}\Big| \leq \sum_{m\geq 0}|\alpha_{nm}|, \quad \forall n\geq 0$$

we have that $\sum_{n\geq 0}\left(\sum_{m\geq 0}\alpha_{nm}\right)$ is absolutely convergent. The same holds true for the series in the right hand side of (3.8). Thus we only need to prove that the two double series are equal.

If N and M are finite natural numbers we have:

$$\sum_{m=0}^{M}\sum_{n=0}^{N}\alpha_{nm} = \sum_{n=0}^{N}\sum_{m=0}^{M}\alpha_{nm}, \tag{3.12}$$

which implies:

$$\sum_{m=0}^{M}\sum_{n\geq 0}\alpha_{nm} - \sum_{n=0}^{N}\sum_{m\geq 0}\alpha_{nm} = \sum_{m=0}^{M}\sum_{n>N}\alpha_{nm} - \sum_{n=0}^{N}\sum_{m>M}\alpha_{nm}, \tag{3.13}$$

which leads to:

$$\left|\sum_{m=0}^{M}\sum_{n\geq 0}\alpha_{nm} - \sum_{n=0}^{N}\sum_{m\geq 0}\alpha_{nm}\right| \leq \sum_{m\geq 0}\sum_{n>N}|\alpha_{nm}| + \sum_{n\geq 0}\sum_{m>M}|\alpha_{nm}|. \tag{3.14}$$

Now we use (3.7) in (3.14): take both M and N to infinity, and obtain:

$$\left|\sum_{m\geq 0}\sum_{n\geq 0}\alpha_{nm} - \sum_{n\geq 0}\sum_{m\geq 0}\alpha_{nm}\right| \leq 0$$

which ends the proof.

□

4 Power series are analytic functions

Let $\{a_n\}_{n\geq 0}\subset\mathbb{R}$ such that $\limsup_{n\to\infty}|a_n|^{1/n}<\infty$. Define $r = 1/\{\limsup_{n\to\infty}|a_n|^{1/n}\}$ if $\limsup_{n\to\infty}|a_n|^{1/n}>0$ and $r=\infty$ if $\limsup_{n\to\infty}|a_n|^{1/n}=0$.

Let $0<R<r$ and define $f:(x_0-R,x_0+R)\mapsto\mathbb{R}$ given by:

$$f(x) := \sum_{n\geq 0}a_n(x-x_0)^n.$$

The series is absolutely convergent because $\limsup_{n\to\infty}|a_n(x-x_0)^n|^{1/n} = \frac{|x-x_0|}{r} < 1$.

Theorem 4.1. *Let $b\in(x_0-R,x_0+R)$ be an arbitrary point. Then f is indefinitely differentiable at b, and for every $t\in(x_0-R,x_0+R)$ with $|t-b|<R-|b-x_0|$ we have:*

$$f(t) = \sum_{m\geq 0}\frac{f^{(m)}(b)}{m!}(t-b)^m,$$

where the Taylor series is absolutely convergent.

Proof. Denote by $\alpha_{nm} := n(n-1)\dots(n-m+1)a_n$ if $m\geq 1$. Note that if $n>k$ we have:

$$(n-k)^{1/n} = \exp(\ln[(n-k)^{1/n}]) = \exp\left(\frac{\ln(n-k)}{n}\right) = \exp\left(\frac{\ln(n)+\ln(1-k/n)}{n}\right)$$

and using (1.4):

$$\exp\left(\frac{\ln(n)}{n} + \frac{\ln(1-k/n)}{n}\right) \to \exp(0) = 1 \quad \text{when} \quad n \to \infty.$$

It follows that

$$\limsup_{n\to\infty} |\alpha_{nm}|^{1/n} = \frac{1}{r}, \quad \forall m \geq 1.$$

Thus the series $\sum_{n\geq m} \alpha_{nm} t^{n-m}$ is absolutely convergent for all $|t| < R$. Given x such that $|x - x_0| \leq \rho < R < r$, there exists some $h_0 > 0$ such that $|x + h - x_0| \leq (R+\rho)/2 < R$ for all $|h| \leq h_0$. Using (1.2) with $a = x - x_0$ and $|h| \leq h_0$ we have:

$$f(x+h) - f(x) = h\sum_{n\geq 1} n a_n (x-x_0)^{n-1} + \frac{h^2}{2}\sum_{n\geq 2} n(n-1)a_n(x + c_{n,a,h} - x_0)^{n-2},$$

where $c_{n,a,h}$ lies between 0 and h. Note that both series on the right hand side converge absolutely because:

$$|na_n(x-x_0)^{n-1}| \leq |\alpha_{n1}|\rho^{n-1}, \quad |n(n-1)a_n(x + c_{n,a,h} - x_0)^{n-2}| \leq |\alpha_{n2}|[(R+\rho)/2]^{n-2}.$$

We conclude that $f'(x) = \sum_{n\geq 1} n a_n (x-x_0)^{n-1}$ for all $|x - x_0| < R$. By induction, we obtain:

$$f^{(m)}(x) = \sum_{n\geq m} \alpha_{nm}(x-x_0)^{n-m}, \quad m \geq 1.$$

It follows that we have the identity:

$$\frac{f^{(m)}(x)}{m!}h^m = \sum_{n\geq m} a_n \frac{n!}{(n-m)!m!} h^m (x-x_0)^{n-m}$$

which holds true for all $m \geq 0$.

Now define $\beta_{nm} = 0$ if $m > n$ and $\beta_{nm} = a_n \frac{n!}{(n-m)!m!} h^m (x-x_0)^{n-m}$ if $m \leq n$. We see that

$$\sum_{m\geq 0} |\beta_{nm}| = \sum_{m=0}^{n} |\beta_{nm}| \leq |a_n| \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} |h|^m |x-x_0|^{n-m} = |a_n|(|h| + |x-x_0|)^n$$

where we used the binomial identity in the last equality. Now if $|h| < R - |x - x_0|$ it follows that $\sum_{n\geq 0} |a_n|(|h| + |x-x_0|)^n < \infty$, hence:

$$\sum_{n\geq 0}\sum_{m\geq 0} |\beta_{nm}| < \infty.$$

The conditions of Theorem 3.1 are satisfied, hence

$$\sum_{n\geq 0}\sum_{m\geq 0} \beta_{nm} = \sum_{m\geq 0}\sum_{n\geq 0} \beta_{nm}.$$

Now we observe that

$$\sum_{n\geq 0}\sum_{m\geq 0} \beta_{nm} = \sum_{n\geq 0}\sum_{m=0}^{n} \beta_{nm} = \sum_{n\geq 0} a_n(x+h-x_0)^n = f(x+h),$$

while

$$\sum_{m\geq 0}\sum_{n\geq 0} \beta_{nm} = \sum_{m\geq 0}\sum_{n\geq m} \beta_{nm} = \sum_{m\geq 0}\sum_{n\geq m} a_n \frac{n!}{(n-m)!m!} h^m (x-x_0)^{n-m} = \sum_{m\geq 0} \frac{f^{(m)}(x)}{m!}h^m.$$

In other words,

$$f(x+h) = \sum_{m\geq 0} \frac{f^{(m)}(x)}{m!}h^m.$$

Now replace $x + h = t$ and $x = b$ and the theorem is proved. $\square$