Typical exam sets

Variant 1

Exercise 1. Consider the power series

$$f(x) := \sum_{n\geq 1} \frac{(x-1)^n}{n}.$$

(i). Find the convergence interval.
(ii). Show that $f'(x) = 1/(2-x)$ at all points where f converges absolutely.
(iii). Prove that $f(x) = -\ln(2-x)$ on the convergence interval. (Hint: use the fundamental theorem of calculus.)

Exercise 2. Consider the sequence $\{f_n\}_{n\geq 1}$ where

$$f_n(x) := \begin{cases} 0 & \text{if } x = 0; \\ \frac{\sin(xn)}{xn} & \text{if } 0 < x \leq 1. \end{cases}$$

(i). Prove that each f_n is discontinuous at $x = 0$ but the sequence has a continuous pointwise limit.
(iii). Does the sequence have a uniform limit?

Exercise 3. Consider the equation

$$y'(t) = y(t)^2 + 1, \quad y(0) = 1.$$

(i). Define $g : \mathbb{R} \mapsto \mathbb{R}$, $g(x) = 1 + x^2$. Let $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, $f(t, x) := g(x)$. Show that $y'(t) = f(t, y(t))$.
(ii). Show that $f \in C^1(\mathbb{R} \times \mathbb{R})$ and it obeys a local Lipschitz condition.
(iii). Show that for t near 0 we can rewrite the equation as:

$$[\arctan(y(t)) - t]' = 0.$$

(iv). Find $y(t)$ and indicate the maximal time interval containing $t_0 = 0$ where the solution exists.

Exercise 4. Let $\mathbf{h} : \mathbb{R}^2 \mapsto \mathbb{R}$ given by $\mathbf{h}(u, v) = u^2 + (v-1)^2 - 5 + e^{u-2}$.
(i). Show that $\mathbf{h}(2, 1) = 0$, and $\mathbf{h} \in C^1(\mathbb{R}^2)$.
(ii). Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $f : (1-\epsilon, 1+\epsilon) \mapsto \mathbb{R}$ such that

$$\mathbf{h}(f(v), v) = 0, \quad \forall v \in (1-\epsilon, 1+\epsilon).$$

(iii). Find $f'(1)$.

Variant 2

Exercise 1. Consider the power series

$$f(x) := \sum_{n\geq 2} \frac{(x+1)^n}{n(n-1)}.$$

(i). Find the convergence interval. Compute $f(0)$.

(ii). Show that $f'(x) = \sum_{n\geq 1} \frac{(x+1)^n}{n}$ and $f''(x) = -1/x$ at all points where f converges absolutely.

(iii). Compute $f(-1)$ and $f'(-1)$. Prove that $f(x) = 1 + x - x\ln(-x)$ on the convergence interval. (Hint: use the fundamental theorem of calculus.)

Exercise 2. Consider the sequence $\{f_n\}_{n\geq 1}$ where

$$f_n(x) := \begin{cases} 1 & \text{if } x = 0; \\ n\sin(x/n) & \text{if } 0 < x \leq 1. \end{cases}$$

(i). Prove that each f_n is discontinuous at $x = 0$.

(ii). Prove that the sequence has a pointwise limit.

(iii). Does the sequence have a uniform limit? (Hint: use the fact the $0 \leq x - n\sin(x/n) \leq 1 - n\sin(1/n)$ if $x \in [0, 1]$).

Exercise 3. Consider the equation

$$y'(t) = (t+1)(y(t)+1), \quad y(0) = 0.$$

(i). Let $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, $f(t, x) := (t+1)(x+1)$. Show that $y'(t) = f(t, y(t))$.

(ii). Show that $f \in C^1(\mathbb{R} \times \mathbb{R})$ and it obeys a local Lipschitz condition.

(iii). Show that for t near 0 we can rewrite the equation as:

$$[\ln(y(t)+1) - (t+1)^2/2]' = 0.$$

(iv). Find $y(t)$ and indicate the maximal time interval containing $t_0 = 0$ where the solution exists.

Exercise 4. Let $\mathbf{h} : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by $\mathbf{h}(u_1, u_2, v_1) = [u_1^2 + (v_1 - 1)^2 - 5 + e^{u_2 - 2}, \ln(v_1 u_1/2)]$.

(i). Show that $\mathbf{h}(2, 2, 1) = [0, 0]$, and $\mathbf{h} \in C^1(\mathbb{R}^3)$.

(ii). Show that one can apply the implicit function theorem in order to obtain some small enough $\epsilon > 0$ and a C^1 function $\mathbf{f} : (1 - \epsilon, 1 + \epsilon) \mapsto \mathbb{R}^2$ such that

$$\mathbf{h}(\mathbf{f}(v_1), v_1) = [0, 0], \quad \forall v_1 \in (1 - \epsilon, 1 + \epsilon).$$

(iii). Find the Jacobi matrix $[D\mathbf{f}](1)$.