On 2D systems of first order linear ODE's

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1 A few things about the exponential function

We will in this section define in a rigorous way and study a few important properties of the exponential function.

Proposition 1.1. *Consider the first order ODE $y'(t) = y(t)$ with the initial condition $y(0) = 1$, and $t \in \mathbb{R}$.*

(i). *There exists a unique solution to this equation, and this solution is called the exponential function e^t.*

(ii). *We have $e^{t+s} = e^t e^s$ for every $t, s \in \mathbb{R}$.*

(iii). *We have $e^t > 0$ for every $t \in \mathbb{R}$, hence the exponential function is increasing.*

(iv). *Consider the sequence of polynomials $p_n(t) := 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$ where $n \geq 1$. We have that $p_n(t) \leq e^t$ for every $t \geq 0$ and*

$$\lim_{n\to\infty} p_n(t) = e^t. \tag{1.1}$$

Proof (i). We will first construct local solutions, and then will show that these solutions can be uniquely extended to the whole real axis by a cut-and-paste technique.

In order to construct a local solution around a given initial point t_0, we will use an argument as in Picard's theorem (see example (3) in Cohen, page 123). Our equation can be put in the form

$$y'(t) = f(t, y(t)), \quad t \in [t_0 - h, t_0 + h], \quad y(t_0) = y_0, \tag{1.2}$$

where $h > 0$, $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y$. The function f obeys a *global* Lipschitz condition, with a constant $K = 1$. Any solution to the above differential equation would also be a solution to the following integral equation (prove it!):

$$y(t) = y_0 + \int_{t_0}^{t} y(s)ds, \quad t \in [t_0 - h, t_0 + h]. \tag{1.3}$$

Consider the complete metric space

$$C([t_0 - h, t_0 + h]) := \{g : [t_0 - h, t_0 + h] \to \mathbb{R} : \quad g \text{ is continuous}\},$$

with the metric $d_\infty : C([t_0 - h, t_0 + h]) \times C([t_0 - h, t_0 + h]) \to \mathbb{R}$,

$$d_\infty(g_1, g_2) := \sup_{t\in[t_0-h,t_0+h]} |g_1(t) - g_2(t)|.$$

Introduce the mapping $A : C([t_0 - h, t_0 + h]) \to C([t_0 - h, t_0 + h])$ given by

$$(A(g))(t) := y_0 + \int_{t_0}^{t} g(s)ds.$$

Now one can see that

$$d_\infty(A(g_1), A(g_2)) = \sup_{t\in[t_0-h,t_0+h]} \left| \int_{t_0}^{t} (g_1(s) - g_2(s))ds \right| \le h d_\infty(g_1, g_2), \quad (1.4)$$

therefore A is a contraction if $h < 1$, *independent* of t_0. Since we can rewrite equation (1.3) as a fixed point equation $A(y) = y$ in a complete metric space, it follows that (1.3) has a unique solution in *any* interval of the type $[t_0 - 1/2, t_0 + 1/2]$ where $y(t_0) = y_0$.

Note: when $y_0 = 0$, then $y \equiv 0$ on $[t_0 - 1/2, t_0 + 1/2]$. This is because all the iterates of the constant function y_0 are zero (verify this!), and that is precisely how we construct the fixed point (as an iteration limit).

Now let us extend this solution to a larger interval. Apply the above local construction to the case in which $t_0 = 0$ and $y_0 = 1$. Denote by $y_1(t)$ the solution of (1.3) on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Apply then the same local construction to the case when $t_0 = \frac{1}{2}$ and $y_0 = y_1(\frac{1}{2})$, and denote by $y_2(t)$ the solution of (1.3) on the interval $[0, 1]$.

The function $m(t) = y_2(t) - y_1(t)$ is well defined on the interval $[0, \frac{1}{2}]$, and obeys $m(\frac{1}{2}) = 0$ and $m'(t) = m(t)$ on the open interval $(0, \frac{1}{2})$. But we know that such an equation with $t_0 = 1/2$ and $y_0 = m(\frac{1}{2}) = 0$ only has an identically zero solution on the interval $[0, 1]$, and in particular on $[0, \frac{1}{2}]$. Thus $y_1(t) = y_2(t)$ on their joint interval $[0, \frac{1}{2}]$. It follows that the function $Y : [-1/2, 1] \to \mathbb{R}$ which is given by $y_1(t)$ on $[-1/2, 1/2)$ and by $y_2(t)$ on $[1/2, 1]$ solves the ODE and obeys the initial condition. Moreover, it is unique, because y_1 is uniquely determined by the initial condition at $t_0 = 0$ and y_2 is uniquely determined by $y_1(\frac{1}{2})$.

We can repeat this procedure in order to extend our solution to larger and larger intervals, and eventually to cover the whole real line. We have therefore shown the existence of a unique global solution to the ODE.

(ii). Let $s \in \mathbb{R}$ be fixed, and consider the ODE $y_s'(t) = y_s(t)$ for $t \in \mathbb{R}$, with the initial condition $y_s(0) = e^s$. Reasoning as before, there exists a unique such solution. Now by direct computation one can verify that both functions e^{t+s} and $e^t e^s$ verify the ODE, therefore they must be equal.

(iii). The exponential function can never be zero. We prove this by contradiction. Assume there exists $s \in \mathbb{R}$ such that $e^s = 0$. Then from *(ii)* we would get $1 = e^0 = e^{-s}e^s = 0$, which is wrong. Thus e^t is a continuous function which is positive at $t = 0$ and can never be zero. Therefore it can neither be negative, because if there was some $s \in \mathbb{R}$ where $e^s < 0$ then due to the intermediate value theorem we would be able to construct a point t_s situated between 0 and s where $e^{t_s} = 0$, and we know that this is not possible.

(iv). If we differentiate the exponential function $n + 1$ times we still get the same function back. Now use the Taylor approximation formula around zero and write

$$e^t = p_n(t) + \frac{t^{n+1}}{(n+1)!} e^{C_{t,n}}, \quad (1.5)$$

where $C_{t,n}$ is some point situated between 0 and t. In any case $|C_{t,n}| \leq |t|$ hence $0 < e^{C_{t,n}} \leq e^{|t|}$ independent of n.

Now if $t \geq 0$, equation (1.5) implies that $p_n(t) \leq e^t$ for every $n \geq 0$. In order to prove the convergence in (1.1) we first need a lemma:

Lemma 1.2. *Let a be a positive real number. Define the sequence $x_n = \frac{a^n}{n!}$. Then $\lim_{n\to\infty} x_n = 0$.*

Proof of the lemma. We have $x_{n+1} = \frac{a}{n+1} x_n$ for any n. Hence the sequence $\{x_n\}_{n\geq[a]}$ is decreasing, and is bounded from below by 0. Thus according to Thm. 1.7.10 in Cohen it must converge to a limit l. Now if we take the limit in the equality $x_{n+1} = \frac{a}{n+1} x_n$ we get (use Thm. 1.7.14 (b) in Cohen) $l = 0 \cdot l = 0$. ☐

We now can prove (1.1). Indeed, from (1.5) we can write:

$$|p_n(t) - e^t| = \left| \frac{t^{n+1}}{(n+1)!} e^{C_{t,n}} \right| \leq \frac{|t|^{n+1}}{(n+1)!} e^{|t|}$$

then we apply Lemma 1.2. The proof of Proposition 1.1 is over. ☐

2 2D systems of ODE's

Consider the ODE:

$$\begin{aligned} x'(t) &= a_{11}x(t) + a_{12}y(t) \\ y'(t) &= a_{21}x(t) + a_{22}y(t), \end{aligned} \tag{2.1}$$

with the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$.

If we introduce $\mathbf{X}(t) = (x(t), y(t))$ and A denotes the linear operator defined by the above matrix $\mathcal{M}(A) = \{a_{ij}\}_{1\leq i,j\leq 2}$, then the ODE can be rewritten as:

$$\mathbf{X}'(t) = A\mathbf{X}(t), \quad \mathbf{X}(t_0) = (x_0, y_0). \tag{2.2}$$

Now A has at least one eigenvalue $\lambda_1 \in \mathbb{C}$ corresponding to an eigenvector $\mathbf{v} = (v_1, v_2)$ (see Thm. 5.10 in Axler). One can easily compute them since λ_1 must solve the equation $(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$. Moreover, because $\mathbf{v} \neq (0, 0)$, we can normalize it and we can assume that $||\mathbf{v}||^2 = |v_1|^2 + |v_2|^2 = 1$.

Now define the vector $\mathbf{u} := (-\overline{v_2}, \overline{v_1})$. Because $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, $\mathbf{u}$ and $\mathbf{v}$ must be linear independent (see Corollary 6.16 in Axler). Hence the list $\{\mathbf{v}, \mathbf{u}\}$ forms an orthonormal basis in $\mathbb{C}^2$, and we can express the unknown vector $\mathbf{X}(t)$ as a linear combination

$$\mathbf{X}(t) = c_1(t)\mathbf{v} + c_2(t)\mathbf{u}, \quad t \in \mathbb{R}. \tag{2.3}$$

The initial values $c_1(0)$ and $c_2(0)$ can be easily obtained using Thm. 6.17 in Axler:

$$c_1(t_0) = \langle \mathbf{X}(t_0), \mathbf{v}\rangle = x_0\overline{v_1} + y_0\overline{v_2}, \quad c_2(t_0) = \langle \mathbf{X}(t_0), \mathbf{u}\rangle = -x_0v_2 + y_0v_1. \tag{2.4}$$

Let us now see what are the equations which must be obeyed by c_1 and c_2. If we differentiate in (2.3) we have:

$$\begin{aligned}\mathbf{X}'(t) &= c_1'(t)\mathbf{v} + c_2'(t)\mathbf{u} \\ &= A\mathbf{X}(t) = A(c_1(t)\mathbf{v} + c_2(t)\mathbf{u}) = c_1(t)A\mathbf{v} + c_2(t)A\mathbf{u} \\ &= c_1(t)\lambda_1\mathbf{v} + c_2(t)A\mathbf{u}. \end{aligned} \tag{2.5}$$

Thus we must have:

$$c_1'(t)\mathbf{v} + c_2'(t)\mathbf{u} = c_1(t)\lambda_1\mathbf{v} + c_2(t)A\mathbf{u}. \tag{2.6}$$

The vector $A\mathbf{u}$ can also be written as a linear combination of $\mathbf{v}$ and $\mathbf{u}$ in the following way (see Thm. 6.17 in Axler):

$$A\mathbf{u} = \langle A\mathbf{u}, \mathbf{v}\rangle\mathbf{v} + \langle A\mathbf{u}, \mathbf{u}\rangle\mathbf{u}.$$

If we denote by $b_{12} := \langle A\mathbf{u}, \mathbf{v}\rangle$ and $\lambda_2 := \langle A\mathbf{u}, \mathbf{u}\rangle$, and if we use this in (2.6) we obtain:

$$c_1'(t)\mathbf{v} + c_2'(t)\mathbf{u} = [\lambda_1 c_1(t) + b_{12}c_2(t)]\mathbf{v} + c_2(t)\lambda_2\mathbf{u}. \tag{2.7}$$

Therefore we must have

$$\begin{aligned} c_1'(t) &= \lambda_1 c_1(t) + b_{12}c_2(t) \\ c_2'(t) &= \lambda_2 c_2(t), \end{aligned} \tag{2.8}$$

with the initial conditions given in (2.4). Now let us find c_1 and c_2.

2.1 Finding c_2

We must solve the equation $c_2'(t) = \lambda_2 c_2(t)$ with the initial condition $c_2(t_0)$. Let us define the function $\phi(t) = e^{-\lambda_2(t-t_0)}c_2(t)$. If we differentiate ϕ we have:

$$\phi'(t) = -\lambda_2 e^{-\lambda_2(t-t_0)}c_2(t) + e^{-\lambda_2(t-t_0)}c_2'(t) = 0,$$

where we used the properties of the exponential function and the equation obeyed by c_2. It means that ϕ is a constant function, which must equal $\phi(t_0) = c_2(t_0)$. Therefore

$$c_2(t) = c_2(t_0)e^{\lambda_2(t-t_0)}.$$

2.2 Finding c_1

We must solve the equation $c_1'(t) = \lambda_1 c_1(t) + b_{12}c_2(t_0)e^{\lambda_2(t-t_0)}$ with the initial condition $c_1(t_0)$. Define the function $\psi(t) = e^{-\lambda_1(t-t_0)}c_1(t)$. Compute its derivative:

$$\psi'(t) = -\lambda_1 e^{-\lambda_1(t-t_0)}c_1(t) + e^{-\lambda_1(t-t_0)}c_1'(t) = b_{12}c_2(t_0)e^{-(\lambda_1-\lambda_2)(t-t_0)},$$

therefore we have:

$$\begin{aligned} \psi'(t) &= b_{12}c_2(t_0)e^{-(\lambda_1-\lambda_2)(t-t_0)}, \\ \psi(t_0) &= c_1(t_0). \end{aligned} \tag{2.9}$$

The fundamental theorem of calculus gives us:

$$\psi(t) = c_1(t_0) + b_{12}c_2(t_0)\int_{t_0}^{t} e^{(\lambda_2-\lambda_1)(s-t_0)}ds. \tag{2.10}$$

If $\lambda_1 = \lambda_2$, we have

$$\psi(t) = c_1(t_0) + b_{12}c_2(t_0)(t - t_0). \tag{2.11}$$

If $\lambda_1 \neq \lambda_2$, we get:

$$\psi(t) = c_1(t_0) + b_{12}c_2(t_0)\frac{e^{(\lambda_2-\lambda_1)(t-t_0)} - 1}{\lambda_2 - \lambda_1}. \tag{2.12}$$

In both cases, $c_1(t) = e^{\lambda_1(t-t_0)}\psi(t)$. Now we can go back to (2.3) and find $x(t)$ and $y(t)$ as:

$$x(t) = c_1(t)v_1 - c_2(t)\overline{v_2}, \quad y(t) = c_1(t)v_2 + c_2(t)\overline{v_1}.$$