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1 Generalized Harper operators

Let $\Gamma \subset \mathbb{Z}^2$ be a lattice (actually it is not necessary to be a periodic lattice, it could be for instance a perturbed, irregular lattice). The Hilbert space is $l^2(\Gamma)$.

The elements of the canonical basis in $l^2(\Gamma)$ are denoted by $\{\delta_{\mathbf{x}}\}_{\mathbf{x}\in\Gamma}$, where $\delta_{\mathbf{x}}(\mathbf{y}) = 1$ if $\mathbf{y} = \mathbf{x}$ and zero otherwise. In the discrete case, to any bounded self-adjoint operator $H \in B(l^2(\Gamma))$ it corresponds a bounded and symmetric integral kernel $H(\mathbf{x}, \mathbf{x}') = \langle H\delta_{\mathbf{x}'}, \delta_{\mathbf{x}}\rangle$.

We introduce the notation $\langle \mathbf{x} - \mathbf{x}_0\rangle^\alpha = [1 + (\mathbf{x} - \mathbf{x}_0)^2]^{\frac{\alpha}{2}}$, $\alpha \geq 0$. We define $\mathcal{C}^\alpha$ to be the set of bounded and self-adjoint operators $H \in B(l^2(\Gamma))$ which have the property that $H(\mathbf{x}, \mathbf{x}')$ obeys a weighted Schur-Holmgren type l^1-estimate:

$$||H||_{\mathcal{C}^\alpha} := \sup_{\mathbf{x}'\in\Gamma} \sum_{\mathbf{x}\in\Gamma} \langle \mathbf{x} - \mathbf{x}'\rangle^\alpha |H(\mathbf{x}, \mathbf{x}')| < \infty. \tag{1.1}$$

We also define the space $\mathcal{H}^\alpha$ which contains bounded and self-adjoint operators H who obey:

$$||H||_{\mathcal{H}^\alpha} := \sup_{\mathbf{x}'\in\Gamma} \left\{ \sum_{\mathbf{x}\in\Gamma} \langle \mathbf{x} - \mathbf{x}'\rangle^{2\alpha} |H(\mathbf{x}, \mathbf{x}')|^2 \right\}^{\frac{1}{2}} < \infty. \tag{1.2}$$

There is a natural embedding given in the following short lemma:

Lemma 1.1. *Let $H \in \mathcal{H}^\alpha$ with $\alpha > 1$. Then $H \in \mathcal{C}^\beta$ with $\beta < \alpha - 1$. In particular, if $\alpha > 3$ then the integral kernel $\langle \mathbf{x} - \mathbf{x}'\rangle^2 |H(\mathbf{x}, \mathbf{x}')|$ obeys a Schur-Holmgren estimate and thus defines a bounded operator.*

Proof. Choose some small enough $\epsilon > 0$ such that $\alpha > \beta + 1 + \epsilon$. We write:

$$\langle \mathbf{x} - \mathbf{x}'\rangle^\beta |H(\mathbf{x}, \mathbf{x}')| \leq \langle \mathbf{x} - \mathbf{x}'\rangle^{-1-\epsilon} \langle \mathbf{x} - \mathbf{x}'\rangle^\alpha |H(\mathbf{x}, \mathbf{x}')|$$

and see that the Cauchy-Schwarz inequality gives

$$||H||_{\mathcal{C}^\beta} \leq C_{\alpha,\beta} ||H||_{\mathcal{H}^\alpha} \ . \tag{1.3}$$

□

We model the magnetic flux generated by a unit magnetic field orthogonal to the plane through a triangle generated by $\mathbf{x}$, $\mathbf{x}'$ and the origin by:

$$\varphi(\mathbf{x}, \mathbf{x}') := -\frac{1}{2}\,(x_1\, x_2' - x_2\, x_1') = -\varphi(\mathbf{x}', \mathbf{x}). \tag{1.4}$$

Note the important additive identity:

$$\begin{aligned} &\varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{y}, \mathbf{x}') = \varphi(\mathbf{x}, \mathbf{x}') + \varphi(\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{x}'), \\ &|\varphi(\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{x}')| \leq \frac{1}{2} |\mathbf{x} - \mathbf{y}|\, |\mathbf{y} - \mathbf{x}'|. \end{aligned} \tag{1.5}$$

Let $K \in \mathcal{C}^\alpha$ with $\alpha \geq 0$. Let its integral kernel be $K(\mathbf{x}, \mathbf{x}')$. We are interested in a family of operators $\{K_b\}_{b\in\mathbb{R}}$ given by the integral kernels $e^{ib\varphi(\mathbf{x},\mathbf{x}')} K(\mathbf{x}, \mathbf{x}')$. Clearly, $\{K_b\}_{b\in\mathbb{R}} \subset \mathcal{C}^\alpha$.

Now let us formulate the main result of our paper.

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Theorem 1.2. *Let $\alpha > 3$. Choose some $K \in \mathcal{H}^\alpha$ and construct the corresponding magnetic operators $\{K_b\}_{b\in\mathbb{R}}$. Then we have:*

i. *The resolvent set $\rho(K_b)$ is stable; more precisely, if $\mathrm{dist}(z, \sigma(K_{b_0})) \geq \epsilon$ then there exists $\delta > 0$ and $\eta > 0$ such that $\mathrm{dist}(z, \sigma(K_b)) \geq \eta$ whenever $|b - b_0| < \delta$.*

ii. *Define $E_+(b) := \sup(\sigma(K_b))$ and $E_-(b) := \inf(\sigma(K_b))$. Then $E_\pm$ are Lipschitz functions of b.*

iii. *Let $\alpha > 4$. Assume that K_{b_0} has a gap in the spectrum of the form $(e_-(b_0), e_+(b_0))$, where $e_\pm(b_0) \in \sigma(K_{b_0})$ are the gap edges. Then as long as the gap is not closing by varying b in a closed interval I containing b_0, the operator K_b will have a gap $(e_-(b), e_+(b))$ whose edges are Lipschitz functions of b on I.*

Remark. Denoting by $\delta b = b - b_0$, according to our notations we have that $K_b = (K_{b_0})_{\delta b}$. It means that it is enough to prove the spectral stability and the Lipschitz properties near $b_0 = 0$.

2 Proof of *(i)*

Let us start by stating a technical result to be proved in the Appendix, which claims that if H has a kernel which is localized near the diagonal, then the resolvent's kernel will also have such a localization. Note that the estimate holds for all $z \in \rho(H)$.

Proposition 2.1. *Let $H \in \mathcal{C}^\alpha$, with $\alpha \geq 0$. Let $z \in \rho(H)$. Then we have $(H - z)^{-1} \in \mathcal{H}^\alpha$, and there exists a constant C independent of z such that*

$$||(H - z)^{-1}||_{\mathcal{H}^\alpha} \leq C \left(\frac{||H||_{\mathcal{C}^\alpha}^{\alpha+1}}{\{\mathrm{dist}(z, \sigma(H))\}^{\alpha+2}} + \frac{1}{\mathrm{dist}(z, \sigma(H))} \right). \tag{2.1}$$

Now let us start the proof of *(i)*. Constants only depending on ϵ will be named C_ϵ even though they might have different values.

Remember that it is enough to prove the stability result near $b_0 = 0$. Let $K \in \mathcal{H}^\alpha$ with $\alpha > 3$. Lemma 1.1 gives us some $\beta > 2$ such that $K \in \mathcal{C}^\beta$. Proposition 2.1 says that $(K - z)^{-1} \in \mathcal{H}^\beta$, while Lemma 1.1 insures that there exists $\gamma > 1$ such that $(K - z)^{-1} \in \mathcal{C}^\gamma$.

Denote by $G(\mathbf{x}, \mathbf{x}'; z)$ the integral kernel of $(K - z)^{-1}$. From (2.1) and (1.3) we obtain a constant C_ϵ such that:

$$\sup_{\mathbf{x}'\in\Gamma} \sum_{\mathbf{x}\in\Gamma} \langle \mathbf{x} - \mathbf{x}' \rangle |G(\mathbf{x}, \mathbf{x}'; z)| \leq C_\epsilon \quad \text{if} \quad \mathrm{dist}(z, \sigma(K)) \geq \epsilon. \tag{2.2}$$

Define the operator $S_b(z)$ to be the one corresponding to the integral kernel $e^{ib\varphi(\mathbf{x},\mathbf{x}')} G(\mathbf{x}, \mathbf{x}'; z)$. Using the Schur-Holmgren criterion we can write

$$||S_b(z)|| \leq C_\epsilon, \quad b \in \mathbb{R}, \quad \mathrm{dist}(z, \sigma(K)) \geq \epsilon.$$

Using (1.5) we can write:

$$(K_b - z)S_b(z) =: 1 + T_b(z), \tag{2.3}$$

where $T_b(z)$ is given by the integral kernel

$$e^{ib\varphi(\mathbf{x},\mathbf{x}')} \sum_{\mathbf{y}\in\Gamma} (e^{ib\varphi(\mathbf{x}-\mathbf{y},\mathbf{x}'-\mathbf{y})} - 1) K_b(\mathbf{x}, \mathbf{y})\, G(\mathbf{y}, \mathbf{x}'; z). \tag{2.4}$$

Note that

$$|e^{ib\varphi(\mathbf{x}-\mathbf{y},\mathbf{x}'-\mathbf{y})} - 1| \leq |b|\, |\varphi(\mathbf{x} - \mathbf{y}, \mathbf{x}' - \mathbf{y})| \leq \frac{|b|}{2}\, |\mathbf{x} - \mathbf{y}|\, |\mathbf{y} - \mathbf{x}'|. \tag{2.5}$$

Then for any $f \in l^2(\Gamma)$ with compact support we can write:

$$|T_b(z)f|(\mathbf{x}) \le |b| \sum_{\mathbf{y}\in\Gamma} |\mathbf{x}-\mathbf{y}|\; |K_b(\mathbf{x},\mathbf{y})|\; |\mathbf{y}-\mathbf{x}'|\; |G(\mathbf{y},\mathbf{x}';z)|\; |f(\mathbf{x}')| \tag{2.6}$$

and after applying the Schur-Holmgren criterion we get:

$$||T_b(z)|| \le |b|\; ||K_b||_{\mathcal{C}^1} ||(K-z)^{-1}||_{\mathcal{C}^1} \le |b|\; C_\epsilon.$$

Thus if $|b|$ is small enough, $||T_b(z)|| \le 1/2$ whenever $\mathrm{dist}(z,\sigma(K)) \ge \epsilon$. From (2.3) we conclude that $K_b - z$ is invertible and there exists a constant C_ϵ such that

$$\begin{aligned}(K_b - z)^{-1} &= S_b(z)\;(1+T_b(z))^{-1},\\ ||(K_b-z)^{-1}|| &\le C_\epsilon \quad \text{whenever } |b| \le b_\epsilon \text{ and } \mathrm{dist}(z,\sigma(K)) \ge \epsilon.\end{aligned} \tag{2.7}$$

This means that $\mathrm{dist}(z,\sigma(K_b)) \ge \frac{1}{C_\epsilon} > 0$ whenever $|b| \le b_\epsilon$ and $\mathrm{dist}(z,\sigma(K)) \ge \epsilon$, and the proof of *(i)* is over. □

3 Proof of *(ii)*

As before, we only need to consider $b_0 = 0$. We give the proof just for the upper spectral limit E_+, since the argument for E_- is similar.

3.1 Reduction to localized operators

The first thing to do is to reduce the problem to operators with kernels supported near the diagonal. If we have two bounded and self adjoint operators A and B, then for any ψ with norm one we can write

$$\langle A\psi,\psi\rangle \le \langle B\psi,\psi\rangle + ||A-B|| \le \sup(\sigma(B)) + ||A-B||$$

which means that $\sup(\sigma(A)) - \sup(\sigma(B)) \le ||A-B||$. By interchanging A with B we obtain the inequality:

$$|\sup(\sigma(A)) - \sup(\sigma(B))| \le ||A-B||. \tag{3.1}$$

Denote by χ the characteristic function of the interval $[0,1]$. Denote by $\hat{K}_b$ the operator given by the integral kernel $\hat{K}_b(\mathbf{x},\mathbf{x}') := \chi\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{b}}\right)\; K(\mathbf{x},\mathbf{x}')$ and by $\tilde{K}_b$ the operator given by $\tilde{K}_b(\mathbf{x},\mathbf{x}') := \chi\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{b}}\right)\; e^{ib\varphi(\mathbf{x},\mathbf{x}')}K(\mathbf{x},\mathbf{x}')$.

Since $K \in \mathcal{H}^\alpha$ with $\alpha > 3$, according to Lemma 1.1 we have the bound:

$$\sup_{\mathbf{x}'\in\Gamma} \sum_{\mathbf{x}\in\Gamma} \langle \mathbf{x}-\mathbf{x}'\rangle^2 |K(\mathbf{x},\mathbf{x}')| = ||K||_{\mathcal{C}^2} < \infty. \tag{3.2}$$

Via the Schur-Holmgren criterion we obtain:

$$\max\{||K-\hat{K}_b||,\; ||K_b-\tilde{K}_b||\} \le \sup_{\mathbf{x}'\in\Gamma}\sum_{\mathbf{x}\in\Gamma}\left[1-\chi\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{b}}\right)\right] |K(\mathbf{x},\mathbf{x}')| \le |b|\; ||K||_{\mathcal{C}^2}. \tag{3.3}$$

Using the triangle inequality and (3.1) we can write:

$$|E_+(b) - E_+(0)| \le 2|b|\; ||K||_{\mathcal{C}^2} + |\sup(\sigma(\tilde{K}_b)) - \sup(\sigma(\hat{K}_b))|. \tag{3.4}$$

Thus we have reduced the problem to the study of the spectral edges of $\tilde{K}_b$ and $\hat{K}_b$.

3.2 A few identities from the continuous case

We list here a few very well known facts about the continuous two dimensional magnetic Schrödinger operator with constant magnetic field equal to b in $L^2(\mathbb{R}^2)$:

$$H_b = (\mathbf{p} - b\mathbf{a}(\mathbf{x}))^2, \quad \mathbf{p} = -i\nabla_{\mathbf{x}}, \quad \mathbf{a}(\mathbf{x}) = (-x_2/2, x_1/2). \tag{3.5}$$

The integral kernel of the semi-group e^{-tH_b} is denoted with $G_b(\mathbf{x}, \mathbf{x}'; t)$ and is given by the following explicit formula:

$$G_b(\mathbf{x}, \mathbf{x}'; t) = e^{ib\varphi(\mathbf{x},\mathbf{x}')} \frac{b}{4\pi \sinh(bt)} \exp\left[-\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(bt)}\right] =: e^{ib\varphi(\mathbf{x},\mathbf{x}')} \tilde{G}_b(\mathbf{x}, \mathbf{x}'; t). \tag{3.6}$$

The semigroup property insures the following identity:

$$G_b(\mathbf{x}, \mathbf{x}'; 2t) = \int_{\mathbf{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y}. \tag{3.7}$$

Then we can write:

$$\begin{aligned} e^{ib\varphi(\mathbf{x},\mathbf{x}')} &= \frac{1}{\tilde{G}_b(\mathbf{x}, \mathbf{x}'; 2t)} \int_{\mathbf{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y} \\ &= \frac{4\pi \sinh(2bt)}{b} \exp\left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(2bt)}\right] \int_{\mathbf{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y}. \end{aligned} \tag{3.8}$$

Taking the complex conjugation in both sides gives:

$$e^{-ib\varphi(\mathbf{x},\mathbf{x}')} = \frac{4\pi \sinh(2bt)}{b} \exp\left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(2bt)}\right] \int_{\mathbf{R}^2} G_b(\mathbf{y}, \mathbf{x}; t) G_b(\mathbf{x}', \mathbf{y}; t) d\mathbf{y}. \tag{3.9}$$

Again the semi-group property gives that:

$$\frac{b}{4\pi \sinh(2bt)} = G_b(\mathbf{x}, \mathbf{x}; 2t) = \int_{\mathbf{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}; t) d\mathbf{y} = \int_{\mathbf{R}^2} |G_b(\mathbf{y}, \mathbf{x}; t)|^2 d\mathbf{y} \tag{3.10}$$

which is clearly $\mathbf{x}$ independent.

3.3 Study of the operators with cut-off

Clearly, $\tilde{K}_b(\mathbf{x}, \mathbf{x}') = e^{ib\varphi(\mathbf{x},\mathbf{x}')} \hat{K}_b(\mathbf{x}, \mathbf{x}')$. Without loss, assume that $b > 0$. Take $\psi \in l^2(\Gamma)$ with compact support and compute (use (3.8) in the second inequality):

$$\begin{aligned} \langle \tilde{K}_b \psi, \psi \rangle &= \sum_{\mathbf{x},\mathbf{x}' \in \Gamma} e^{ib\varphi(\mathbf{x},\mathbf{x}')} \hat{K}_b(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \\ &= \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x},\mathbf{x}' \in \Gamma} \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \frac{4\pi \sinh(2bt)}{b} \hat{K}_b(\mathbf{x}, \mathbf{x}') \exp\left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(2bt)}\right] G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t). \end{aligned} \tag{3.11}$$

Now denote by $A_b(t)$ the operator with kernel

$$A_b(\mathbf{x}, \mathbf{x}'; t) := \hat{K}_b(\mathbf{x}, \mathbf{x}') \exp\left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(2bt)}\right] = K(\mathbf{x}, \mathbf{x}') \exp\left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4\tanh(2bt)}\right] \chi\left(\frac{|\mathbf{x} - \mathbf{x}'|}{\sqrt{b}}\right).$$

The crucial observation is that equation (3.11) leads to:

$$\begin{aligned} \langle \tilde{K}_b \psi, \psi \rangle &= \int_{\mathbf{R}^2} d\mathbf{y} \langle A_b(t) G_b(\mathbf{y}, \cdot; t) \psi, G_b(\mathbf{y}, \cdot; t) \psi \rangle \frac{4\pi \sinh(2bt)}{b} \\ &\le \sup \sigma(A_b(t)) \frac{4\pi \sinh(2bt)}{b} \int_{\mathbf{R}^2} d\mathbf{y} ||G_b(\mathbf{y}, \cdot; t) \psi||^2 \\ &= \sup \sigma(A_b(t)) \frac{4\pi \sinh(2bt)}{b} \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x} \in \Gamma} |G_b(\mathbf{y}, \mathbf{x}; t)|^2 |\psi(\mathbf{x})|^2 \\ &= \sup \sigma(A_b(t)) \; ||\psi||^2, \end{aligned} \tag{3.12}$$

where in the last line we used (3.10). It means that $\sup\sigma(\tilde{K}_b) \leq \sup(\sigma(A_b(t)))$ for all t. Now let us show that the operator $A_b(t) - \hat{K}_b$ has a norm proportional with b if t is large enough (say $t = b^{-1}$). Indeed, we can write

$$
\begin{aligned}
&|A_b(\mathbf{x},\mathbf{x}';b^{-1}) - \hat{K}_b(\mathbf{x},\mathbf{x}')| \leq |K(\mathbf{x},\mathbf{x}')|\chi\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{b}}\right)\left(\exp\left[\frac{b|\mathbf{x}-\mathbf{x}'|^2}{4\tanh(2)}\right] - 1\right) \\
&\leq |K(\mathbf{x},\mathbf{x}')|\chi\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{b}}\right)\frac{b|\mathbf{x}-\mathbf{x}'|^2}{4\tanh(2)}\exp\left[\frac{b|\mathbf{x}-\mathbf{x}'|^2}{4\tanh(2)}\right] \qquad (3.13)
\end{aligned}
$$

and on the support of χ we can bound the above difference with:

$$
|A_b(\mathbf{x},\mathbf{x}';b^{-1}) - \hat{K}_b(\mathbf{x},\mathbf{x}')| \leq \text{const}\; b\; |\mathbf{x}-\mathbf{x}'|^2 |K(\mathbf{x},\mathbf{x}')|. \qquad (3.14)
$$

The right hand side defines an operator whose norm behaves like b. Thus (3.12) and (3.14) imply:

$$
\sup\sigma(\tilde{K}_b) \leq \sup\sigma(A_b(b^{-1})) \quad \text{and} \quad ||A_b(b^{-1}) - \hat{K}_b|| \leq C\, b. \qquad (3.15)
$$

Using (3.1) we arrive at:

$$
\sup\sigma(\tilde{K}_b) \leq \sup\sigma(\hat{K}_b) + C\, b. \qquad (3.16)
$$

We now want to interchange $\tilde{K}_b$ and $\hat{K}_b$ in the above inequality, which would lead to $\sup\sigma(\hat{K}_b) \leq \sup\sigma(\tilde{K}_b) + C\, b$ and thus:

$$
|\sup\sigma(\tilde{K}_b) - \sup\sigma(\hat{K}_b)| \leq C\, b,
$$

which together with (3.4) would imply:

$$
|E_+(b) - E_+(0)| \leq C\, b, \quad b \geq 0.
$$

The key step in the proof of (3.16) was (3.11). Since $\hat{K}_b(\mathbf{x},\mathbf{x}') = e^{-ib\varphi(\mathbf{x},\mathbf{x}')}\tilde{K}_b(\mathbf{x},\mathbf{x}')$ we can write (use (3.9) in the second line):

$$
\begin{aligned}
\langle \hat{K}_b\psi,\psi\rangle &= \sum_{\mathbf{x},\mathbf{x}'\in\Gamma} e^{-ib\varphi(\mathbf{x},\mathbf{x}')}\tilde{K}_b(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')\overline{\psi(\mathbf{x})} \\
&= \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x},\mathbf{x}'\in\Gamma} \psi(\mathbf{x}')\overline{\psi(\mathbf{x})}\frac{4\pi\sinh(2bt)}{b}\tilde{K}_b(\mathbf{x},\mathbf{x}')\exp\left[\frac{b|\mathbf{x}-\mathbf{x}'|^2}{4\tanh(2bt)}\right] G_b(\mathbf{x}',\mathbf{y};t)G_b(\mathbf{y},\mathbf{x};t). \qquad (3.17)
\end{aligned}
$$

Now everything will work as before, because the phase $e^{ib\varphi(\mathbf{x},\mathbf{x}')}$ changes neither the localization nor the $\mathcal{C}^2$ norm of the operators. The proof for the upper spectral edges is over.

The proof for the lower spectral edges is based on an estimate which is very similar with (3.12), in which we reverse the inequality and show that $\inf\sigma(\tilde{K}_b) \geq \inf\sigma(A_b(t))$ for all t. We give no further details.

4 Proof of *(iii)*

The idea is to reduce the problem to the previous case. Again it is enough to consider $b_0 = 0$ and $b > 0$ small enough. Assume that K has a gap in its spectrum of the form (e_-, e_+), with $e_\pm \in \sigma(K)$. Then due to *(i)* we know that if b is small enough the gap will survive: we can choose a positively oriented circle L in the complex plane containing $\Sigma_+(b) := \sigma(K_b) \cap (e_+(b), \infty)$ such that

$$
\text{dist}(z, \sigma(K_b)) \geq \eta > 0 \quad \text{whenever} \quad z \in L \quad \text{and} \quad 0 < b < b_\eta.
$$

The orthogonal projector P_b corresponding to $\Sigma_+(b)$ can be written as a Riesz integral and we have:

$$P_b := \frac{i}{2\pi}\int_L (K_b - z)^{-1}dz, \quad K_bP_b = \frac{i}{2\pi}\int_L z(K_b-z)^{-1}dz, \quad b \geq 0. \tag{4.1}$$

If we consider K_bP_b as an operator living on the whole space $l^2(\Gamma)$, then its spectrum is given by the union $\{0\} \cup \Sigma_+(b)$. If we choose $\lambda := 1 + \sup\sigma(K)$, then for b small enough the operator $D_b := K_bP_b - \lambda P_b$ will have $\inf\sigma(A_b) = e_+(b) - \lambda \leq -1/2$. Thus $e_+(b) = \lambda + \inf\sigma(A_b)$, hence $e_+(b)$ is Lipschitz at $b = 0$ if $\inf\sigma(A_b)$ has the same property. This is what we prove next:

Lemma 4.1. *Let $D_b = K_bP_b - \lambda P_b$ with $\lambda := 1+\sup\sigma(K)$. Then there exists $b_1 > 0$ small enough and a constant $C > 0$ such that for every $0 < b < b_1$ we have $|\inf\sigma(D_b) - \inf\sigma(D_0)| \leq C\, b$.*

Proof. Remember that we imposed $\alpha > 4$. We have that $||K_b||_{\mathcal{H}^\alpha} = ||K||_{\mathcal{H}^\alpha} < \infty$ for all b. According to Lemma 1.1, there exists $\beta > 3$ such that $||K_b||_{\mathcal{C}^\beta} = ||K||_{\mathcal{C}^\beta} < \infty$. Then if b is smaller than some constant only depending on L, Proposition 2.1 tells us $(K_b - z)^{-1} \in \mathcal{H}^\beta$ for all $z \in L$ and $\sup_{z\in L}||(K_b - z)^{-1}||_{\mathcal{H}^\beta} \leq C$. Thus both P_b and D_b belong to $\mathcal{H}^\beta$ if b is small enough. More precisely, there exists $b_2 > 0$ sufficiently small such that

$$\max\{||P_b||_{\mathcal{H}^\beta}, ||D_b||_{\mathcal{H}^\beta}\} \leq C, \quad 0 \leq b \leq b_2. \tag{4.2}$$

If $G(\mathbf{x},\mathbf{x}';z)$ is the integral kernel of $(K-z)^{-1}$, then we introduced at point *(i)* the operator $S_b(z)$ given by the kernel $e^{ib\varphi(\mathbf{x},\mathbf{x}')}G(\mathbf{x},\mathbf{x}';z)$. Using (2.7) we can write:

$$\sup_{z\in L}||(K_b - z)^{-1} - S_b(z)|| \leq C\, b, \tag{4.3}$$

provided b is small enough. Denoting by D_0 the operator given by the integral kernel

$$D_0(\mathbf{x},\mathbf{x}') := \frac{i}{2\pi}\int_L (z-\lambda)G(\mathbf{x},\mathbf{x}';z)dz$$

and by $(D_0)_b$ the operator generated by $e^{ib\varphi(\mathbf{x},\mathbf{x}')}D_0(\mathbf{x},\mathbf{x}')$, then using (4.3) we arrive at the estimate:

$$||D_b - (D_0)_b|| \leq C\, b \quad \text{whenever} \quad 0 \leq b < b_2. \tag{4.4}$$

It follows that $\inf\sigma(D_b)$ is Lipschitz at $b = 0$ if $\inf\sigma((D_0)_b)$ has the same property. But for the operator $(D_0)_b$ we can apply point *(ii)*, and the proof is over.

□

5 Appendix: proof of Proposition 2.1

Denote by $G(\mathbf{x},\mathbf{x}';z)$ the integral kernel of $(H-z)^{-1}$. If $\alpha = 0$ we have

$$\sum_{\mathbf{x}\in\Gamma}|G(\mathbf{x},\mathbf{x}';z)|^2 = ||(H-z)^{-1}\delta_{\mathbf{x}'}||^2 \leq \frac{1}{\{\mathrm{dist}(z,\sigma(H))\}^2}$$

uniformly in $\mathbf{x}'$, an estimate which is in fact much better than (2.1). So from now on we may assume that $\alpha > 0$.

For $\mathbf{k} \in \mathbb{R}^2$ define the unitary multiplication operator $U_\mathbf{k}$ by $(U_\mathbf{k}f)(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{x})$. Define the family of isospectral operators $H_\mathbf{k} = U_\mathbf{k}HU_\mathbf{k}^*$, with integral kernels given by $H_\mathbf{k}(\mathbf{x},\mathbf{x}') = e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}H(\mathbf{x},\mathbf{x}')$. We need the following technical result:

Lemma 5.1. *Let H be an element of $\mathcal{C}^\alpha$. Let n be the integer part of α. Then the mapping*

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto H_\mathbf{k} \in B(l^2(\Gamma))$$

is n times continuously differentiable in the norm topology. Moreover, any n'th order mixed partial derivative of $H_\mathbf{k}$ is $\alpha - n$ Hölder continuous at $\mathbf{k} = 0$ in the norm topology.

Proof. Assume that $\mathbf{k} = (k_1, k_2)$. The integral kernel of $H_{\mathbf{k}}$ is $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}H(\mathbf{x},\mathbf{x}')$. Let n be the integer part of α. Then $H_{\mathbf{k}}$ is n times differentiable in the norm topology with respect to k_j, $j \in \{1,2\}$, and its n'th mixed partial derivative $\partial_{k_1}^m \partial_{k_2}^{n-m} H_{\mathbf{k}}$ is given by the integral kernel $i^n(x_1 - x_1')^m (x_2 - x_2')^{n-m} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}H(\mathbf{x},\mathbf{x}')$. This integral kernel defines a bounded operator because $|(x_1 - x_1')^m (x_2 - x_2')^{n-m}| \leq \langle \mathbf{x} - \mathbf{x}' \rangle^n$ and then we can use (1.1).

For the Hölder continuity statement, we use the estimate $|e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} - 1| \leq 2^{1-\beta}\, |\mathbf{k}|^\beta |\mathbf{x} - \mathbf{x}'|^\beta$ which holds for every $0 \leq \beta \leq 1$.

□

Now let $z \in \rho(H)$. Denote by $G_{\mathbf{k}}(\mathbf{x},\mathbf{x}';z)$ the integral kernel of $(H_{\mathbf{k}} - z)^{-1}$. Due to the identity $U_{\mathbf{k}}(H - z)^{-1}U_{\mathbf{k}}^* = (H_{\mathbf{k}} - z)^{-1}$ we have:

$$G_{\mathbf{k}}(\mathbf{x},\mathbf{x}';z) = e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}G(\mathbf{x},\mathbf{x}';z). \tag{5.1}$$

Let us denote by n the integer part of α. We can suppose that $n \geq 1$ since the case $0 < \alpha < 1$ is covered by the argument below.

From the identity

$$(H_{\mathbf{k}'} - z)^{-1} - (H_{\mathbf{k}} - z)^{-1} = -(H_{\mathbf{k}'} - z)^{-1}[H_{\mathbf{k}'} - H_{\mathbf{k}}](H_{\mathbf{k}} - z)^{-1} \tag{5.2}$$

and from Lemma 5.1 we conclude that the map

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto (H_{\mathbf{k}} - z)^{-1} \in B(l^2(\Gamma))$$

is continuous in the norm topology, and also differentiable. We have:

$$D_{\mathbf{k}}(H_{\mathbf{k}} - z)^{-1} = -(H_{\mathbf{k}} - z)^{-1}[D_{\mathbf{k}}H_{\mathbf{k}}](H_{\mathbf{k}} - z)^{-1}. \tag{5.3}$$

Using this identity at $\mathbf{k} = 0$ in (5.1) leads to:

$$(\mathbf{x} - \mathbf{x}')G(\mathbf{x},\mathbf{x}';z) = -\langle (H - z)^{-1}[D_{\mathbf{k}}H_{\mathbf{k}}]_{\mathbf{k}=0}(H - z)^{-1}\delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle$$

which gives:

$$||(H - z)^{-1}||_{\mathcal{H}^1} \leq C\left(\frac{1}{\mathrm{dist}(z,\sigma(H))^2}||H||_{\mathcal{C}^1} + \frac{1}{\mathrm{dist}(z,\sigma(H))}\right).$$

This is true because we have the pointwise bound

$$\begin{aligned}\langle \mathbf{x} - \mathbf{x}' \rangle^\alpha &\leq (1 + |x_1 - x_1'| + |x_2 - x_2'|)^\alpha \leq (3\max\{1, |x_1 - x_1'|, |x_2 - x_2'|)^\alpha \\ &\leq 3^\alpha + \sum_{j=1}^{2} 3^\alpha |x_j - x_j'|^\alpha. \end{aligned}\tag{5.4}$$

By induction we obtain the following rough estimate:

$$||(H - z)^{-1}||_{\mathcal{H}^n} \leq C_n\left(\frac{1}{\mathrm{dist}(z,\sigma(H))^{n+1}}||H||_{\mathcal{C}_n}^n + \frac{1}{\mathrm{dist}(z,\sigma(H))}\right). \tag{5.5}$$

Now let us assume that $n < \alpha < n + 1$. The integral kernel of the n'th partial derivative of $(H_{\mathbf{k}} - z)^{-1}$ with respect to k_1 is given by $i^n e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}(x_1 - x_1')^n G(\mathbf{x},\mathbf{x}';z)$. Moreover, using (5.3) and Lemma 5.1 we conclude that the operator $\partial_{k_1}^n(H_{\mathbf{k}} - z)^{-1}$ is $\alpha - n$ Hölder continuous at $\mathbf{k} = 0$. Let $\mathbf{k} = (k_1, 0)$. We also have the identity:

$$i^n(e^{ik_1(x_1 - x_1')} - 1)(x_1 - x_1')^n G(\mathbf{x},\mathbf{x}';z) = \langle [\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}]\delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle.$$

The following norm estimate holds true according to Lemma 5.1:

$$||[\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}]|| \leq C|k_1|^{\alpha-n}\left(\frac{||H||_{\mathcal{C}^\alpha}^{n+1}}{\mathrm{dist}(z,\sigma(H))^{n+2}} + \frac{1}{\mathrm{dist}(z,\sigma(H))}\right). \tag{5.6}$$

It means that the following integral converges in norm and defines a bounded operator:

$$\tilde{H} := \int_0^\infty \frac{1}{k_1^{1+\alpha-n}} [\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}] dk_1.$$

Its integral kernel is given by

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; z) := i^n (x_1 - x_1')^n G(\mathbf{x}, \mathbf{x}'; z) \int_0^\infty \frac{1}{k_1^{1+\alpha-n}} (e^{ik_1(x_1-x_1')} - 1) dk_1.$$

Assuming without loss of generality that $x_1 - x_1' \neq 0$, and by a change of variable $s = k_1/|x_1 - x_1'|$ we obtain:

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; z) = |x_1 - x_1'|^{\alpha-n} (x_1 - x_1')^n G(\mathbf{x}, \mathbf{x}'; z) \int_0^\infty \frac{1}{s^{1+\alpha-n}} i^n (e^{is\ \mathrm{sign}(x_1-x_1')} - 1) ds.$$

Notice that the above integral only has two possible values $C_\pm$ both different from zero, depending on the sign of $x_1 - x_1$. Since $\tilde{G}(\mathbf{x}, \mathbf{x}'; z) = \langle \tilde{H} \delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle = C_\pm(x_1, x_1')\ |x_1 - x_1'|^{\alpha-n} (x_1 - x_1')^n G(\mathbf{x}, \mathbf{x}'; z)$ with $|C_\pm(x_1, x_1')| \geq C$ it follows that

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |x_1 - x_1'|^{2\alpha}\ |G(\mathbf{x}, \mathbf{x}'; z)|^2 \leq C^{-2} ||\tilde{H}||^2.$$

This argument can be repeated for the other coordinate and bound the l^2 norm of $\langle \cdot - \mathbf{x}' \rangle^\alpha G(\cdot, \mathbf{x}'; z)$ using (5.4). The proof of Proposition 2.1 is over.