Continuity and connectedness

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20/11/2006

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1 Open sets and continuity

Definition 1.1. *Let (X, d) a metric space. The open ball with radius r and center at x is the set $B_r(x) := \{y \in X : \ d(y, x) < r\}$.*

A set $\mathcal{O} \subseteq X$ is said to be open if either one of the three situations occurs: 1. $\mathcal{O} = \emptyset$; 2. $\mathcal{O} = X$; 3. For every $x \in O$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{O}$.

A set S is said to be closed in X if $X \setminus S$ is open.

Example: let $X = \{4, 6\} \cup [9, 22)$, with the Euclidean distance. Then $B_1(4) = \{4\}$, $B_5(4) = \{4, 6\}$, $B_4(9) = \{6\} \cup [9, 13)$.

Definition 1.2. *Let (X, d_1) and (Y, d_2) be two metric spaces, and consider the function $f : X \to Y$. We say that f is continuous at $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0) > 0$ such that*

$$f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0)). \tag{1.1}$$

We say that f is continuous on X if it is continuous at every point $x \in X$.

Note that (1.1) is the same as the "usual" $\epsilon - \delta$ definition which states:

$$d_2(f(x), f(x_0)) < \epsilon \quad \text{whenever} \quad d_1(x, x_0) < \delta. \tag{1.2}$$

For example, take $X = \{3\} \cup (5, 9]$ and $Y = \mathbb{R}$, both with the Euclidean distance. Define $f : X \to Y$, $f(x) = x + 1$. Let us prove that f is continuous at $x_0 = 3$. Indeed, fix some $\epsilon > 0$. Then $f(x_0) = 4$, and $4 \in B_\epsilon(f(x_0))$. If we choose any $\delta \in (0, 2]$ we have $B_\delta(3) = \{3\}$, hence

$$f(B_\delta(x_0)) = \{4\} \subseteq B_\epsilon(f(x_0)).$$

Definition 1.3. *Let (X, d_1) and (Y, d_2) be two metric spaces, and consider the function $f : X \to Y$. We say that f is sequentially continuous at $x_0 \in X$ if for every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$ we also have that $\lim_{n\to\infty} f(x_n) = f(x_0)$. We say that f is sequentially continuous on X if it is sequentially continuous at every point $x \in X$.*

Lemma 1.4. *The function $f : X \to Y$ is continuous on X if and only if it is sequentially continuous on X.*

Proof. Assume that f is continuous at $x_0 \in X$. Take any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\lim_{n\to\infty} x_n = x_0$. We want to prove that $\lim_{n\to\infty} f(x_n) = f(x_0)$. Indeed, fix some $\epsilon > 0$. According to Definition 1.1, we obtain a δ as in (1.1). But there exists $N \geq 1$ such that $x_n \in B_\delta(x_0)$ whenever $n \geq N$. Using (1.1), it means that $f(x_n) \in B_\epsilon(f(x_0))$ or $d_2(f(x_n), f(x_0)) < \epsilon$ whenever $n \geq N$, hence $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Now assume that f is sequentially continuous at x_0, but not continuous. Not being continuous means that there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ we have

$$f(B_\delta(x_0)) \setminus B_{\epsilon_0}(f(x_0)) \neq \emptyset. \tag{1.3}$$

In other words, if instead of δ we take $1/k$ for every $k \geq 1$, then we obtain a point $x_k \in B_{\frac{1}{k}}(x_0)$ such that $f(x_k) \notin B_{\epsilon_0}(f(x_0))$, or $d_2(f(x_k), f(x_0)) \geq \epsilon_0$ for all k. But this means that we constructed a sequence $\{x_k\}_{k\geq 1}$ which converges to x_0, while $\{f(x_k)\}_{k\geq 1}$ does not converge to $f(x_0)$, and this is a contradiction with our hypothesis. □

We now give the main result of this section. A function is continuous if and only if "it returns open sets into open sets".

Theorem 1.5. *Consider a function $f : X \to Y$ between two metric spaces. The following two statements are equivalent:*

P1. f is continuous on X;

P2. For every open set $V \subset Y$, we have that the preimage

$$f^{-1}(V) := \{x \in X : \ f(x) \in V\}$$

is open in X.

Proof. Let us show that P1 implies P2. Choose some open set V in Y. If $f^{-1}(V) = \emptyset$, then it is open. If $f^{-1}(V) \neq \emptyset$, then consider an arbitrary point $x_0 \in f^{-1}(V)$. We will construct an open ball centered at x_0 which is completely contained in $f^{-1}(V)$, thus showing that x_0 is an interior point. Indeed, because $f(x_0) \in V$, and since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x_0)) \subset V$. Now since f is continuous at x_0, there exists $\delta > 0$ such that (1.1) holds, thus

$$f(B_\delta(x_0)) \subset B_\epsilon(f(x_0)) \subset V,$$

which precisely means that $B_\delta(x_0) \subset f^{-1}(V)$.

We now show that P2 implies P1. Let $x_0 \in X$ and $f(x_0) \in Y$. The open ball $B_\epsilon(f(x_0))$ is an open set in Y (show it!), hence $f^{-1}(B_\epsilon(f(x_0)))$ must be open (and nonempty, since it contains x_0). Thus x_0 is an interior point of this set, thus we can construct $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$, which is the same as (1.1), and we are done. □

2 Connectedness

Definition 2.1. *Let (X, d) a metric space and let $S \subset X$. We say that S is connected if one cannot find two nonempty open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in X such that*

C1. $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$;
C2. $S \subset \mathcal{O}_1 \cup \mathcal{O}_2, \quad S \cap \mathcal{O}_1 \neq \emptyset, \quad S \cap \mathcal{O}_2 \neq \emptyset.$

Examples: let $X = \{3, 6\} \cup (9, 12]$ with the Euclidean metric. The set $S = \{3, 6\}$ is not connected, because we can find two nonempty open sets $\mathcal{O}_1 = \{3\}$ and $\mathcal{O}_2 = \{6\}$ which obey C1 and C2. The set $(9, 12]$ is connected.

Theorem 2.2. *Consider a continuous function $f : X \to Y$ between two metric spaces. If S is a connected set in X, then $f(S)$ is connected in Y.*

Proof. Assume that we can find $\mathcal{O}_1$ and $\mathcal{O}_2$ in Y, both open and nonempty, disjoint, which cover $f(S)$, and both having common points with $f(S)$. Thus we have that $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_2)$ are not empty, and their intersections with S are not empty. Second, they are open due to Theorem 1.5. Third,

$$S \subset f^{-1}(\mathcal{O}_1) \cup f^{-1}(\mathcal{O}_2). \tag{2.1}$$

In order to see this, we will show that every $x \in S$ must belong to the union at the right hand side. Indeed, $f(x) \in f(S)$. Since $f(S) \subset \mathcal{O}_1 \cup \mathcal{O}_2$, then $f(x)$ must belong either to $\mathcal{O}_1$ or to $\mathcal{O}_2$. But this means that x either belongs to $f^{-1}(\mathcal{O}_1)$ or to $f^{-1}(\mathcal{O}_2)$.

Finally, $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_2)$ are disjoint. Indeed, if $x \in f^{-1}(\mathcal{O}_1) \cap f^{-1}(\mathcal{O}_2)$ then $f(x) \in \mathcal{O}_1$ and $f(x) \in \mathcal{O}_2$ which would contradict the hypothesis $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.

Therefore $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_1)$ obey C1 and C2, contradicting the hypothesis that S is connected. $\square$