Notes for the course *Operatorer i Hilbertrum*

Horia Cornean, d. 27/03/2006.

1 Compact (kompakt) and sequentially compact (følgekompakt) sets

Definition 1.1. *Let A be a subset of a metric space (X, d). Let $\mathcal{F}$ be an arbitrary set of indices, and consider the family of sets $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$, where each $\mathcal{O}_\alpha \subseteq X$ is open. This family is called an open covering of A if $A \subseteq \bigcup_{\alpha\in\mathcal{F}} \mathcal{O}_\alpha$.*

Definition 1.2. *Assume that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}}$ is an open covering of A. If $\mathcal{F}'$ is a subset of $\mathcal{F}$, we say that $\{\mathcal{O}_\alpha\}_{\alpha\in\mathcal{F}'}$ is a subcovering if we still have the property $A \subseteq \bigcup_{\alpha\in\mathcal{F}'} \mathcal{O}_\alpha$. A subcovering is called finite, if $\mathcal{F}'$ contains finitely many elements.*

Definition 1.3. *Let A be a subset of a metric space (X, d). Then we say that A is covered by a finite ϵ-net if there exists a natural number $N_\epsilon < \infty$ and the points $\{\mathbf{x}_1, ..., \mathbf{x}_{N_\epsilon}\} \subseteq A$ such that $A \subseteq \bigcup_{j=1}^{N_\epsilon} B_\epsilon(\mathbf{x}_j)$.*

Definition 1.4. *A subset $A \subset X$ is called compact, if from* ANY *open covering of A one can extract a* FINITE *subcovering.*

Definition 1.5. *$A \subset X$ is called sequentially compact if from any sequence $\{x_n\}_{n\geq 1} \subseteq A$ one can extract a subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to some point $x_\infty \in A$.*

We will see that in metric spaces the two notions of compactness are equivalent.

1.1 Compact implies sequentially compact

Theorem 1.6. *Assume that $A \subseteq X$ is compact. Then A is sequentially compact.*

Proof. Assume that there exists a sequence $\{x_n\}_{n\geq 1}$ with no convergent subsequence in A. Such a sequence must have an infinite number of distinct points (exercise). To give a hint, assume that the range of this sequence is $\{a, b\}$. If there only exist a finite number of points in the sequence which are equal with a, then there must exist an infinite number of points which are equal with b. These points would thus define a convergent subsequence, contradicting our hypothesis.

Therefore we can assume that $\{x_n\}_{n\geq 1}$ has no accumulation points in A (otherwise such a point would be the limit of a subsequence). Now choose an arbitrary point $x \in A$. Because x is not an accumulation point for $\{x_n\}_{n\geq 1}$, there exists $\epsilon_x > 0$ such that the ball $B_{\epsilon_x}(x)$ contains at most one element of $\{x_n\}_{n\geq 1}$.

Because $\{B_{\epsilon_x}(x)\}_{x\in A}$ is an open covering for A, and since A is compact, we can extract a finite subcovering from it:

$$A \subseteq \bigcup_{j=1}^{N} B_{\epsilon_{x_j}}(x_j), \quad N < \infty.$$

But since $\{x_n\}_{n\geq 1} \subseteq A$, and because we know that there are at most N distinct points of this sequence in the union $\bigcup_{j=1}^N B_{\epsilon_{x_j}}(x_j)$, we conclude that $\{x_n\}_{n\geq 1}$ can only have a finite number of distinct points, thus it must admit a convergent subsequence. This contradicts our hypothesis. $\square$

1.2 Sequentially compact implies compact

The proof of this fact is slightly more complicated. We need a preparatory result:

Proposition 1.7. *Let A be a sequentially compact set. Then for every $\epsilon > 0$, A can be covered by a finite ϵ-net (see Definition 1.3).*

Proof. If A contains finitely many points, then the proof is obvious. Thus we assume $\#(A) = \infty$.

Now assume that there exists some $\epsilon_0 > 0$ such that A cannot be covered by a finite ϵ_0-net. This means that for any N points of A, $\{x_1, ..., x_N\}$, we have:

$$A \not\subset \bigcup_{j=1}^N B_{\epsilon_0}(x_j). \tag{1.1}$$

We will now construct a sequence with elements in A which cannot have a convergent subsequence. Choose an arbitrary point $x_1 \in A$. We know from (1.1), for $N = 1$, that we can find $x_2 \in A$ such that $x_2 \in A \setminus B(x_1, \epsilon_0)$. This means that $d(x_1, x_2) \geq \epsilon_0$. We use (1.1) again, for $N = 2$, in order to get a point $x_3 \in A \setminus [B(x_1, \epsilon_0) \cup B(x_2, \epsilon_0)]$. This means that $d(x_3, x_1) \geq \epsilon_0$ and $d(x_3, x_2) \geq \epsilon_0$. Thus we can continue with this procedure and construct a sequence $\{x_n\}_{n\geq 1} \subseteq A$ which obeys

$$d(x_j, x_k) \geq \epsilon_0, \quad j \neq k.$$

In other words, we constructed a sequence in A which consists only from isolated points, and which cannot have a convergent subsequence. This contradicts Definition 1.5. $\square$

Let us now prove the theorem:

Theorem 1.8. *Assume that $A \subseteq X$ is sequentially compact. Then A is compact.*

Proof. Consider an arbitrary open covering of A:

$$A \subseteq \bigcup_{\alpha\in\mathcal{F}} \mathcal{O}_\alpha.$$

We will show that we can extract a finite subcovering from it.

For every $x \in A$, there exists at least one open set $\mathcal{O}_{\alpha(x)}$ such that $x \in \mathcal{O}_{\alpha(x)}$. Because $\mathcal{O}_{\alpha(x)}$ is open, we can find $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathcal{O}_{\alpha(x)}$.

For a fixed x, we consider the supremum over all radii $\epsilon > 0$ which obey the condition that there exists at least one $\alpha \in \mathcal{F}$ such that $B_\epsilon(x) \subseteq \mathcal{O}_\alpha$. This supremum is larger than zero, since there exists at least one positive such ϵ. Now write this supremum as $2\epsilon_x > 0$. It means that if we take $\epsilon' > 2\epsilon_x$, then for every $\alpha \in \mathcal{F}$ we have $B_{\epsilon'}(x) \not\subseteq \mathcal{O}_\alpha$.

Let us write an important relation:

$$A \subseteq \bigcup_{x\in A} B_{\epsilon_x}(x) \subseteq \bigcup_{\alpha\in\mathcal{F}} \mathcal{O}_\alpha. \tag{1.2}$$

The first inclusion is obvious, while the second one follows from the above discussion.

We now need to prove a lemma:

Lemma 1.9. *If A is sequentially compact, then*

$$\inf_{x\in A} \epsilon_x =: 2\epsilon_0 > 0.$$

In other words, there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq B_{\epsilon_x}(x)$, for every $x \in A$.

Proof. Assume that $\inf_{x\in A} \epsilon_x = 0$. This implies that there exists a sequence $\{x_n\}_{n\geq 1} \subseteq A$ such that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. Since A is sequentially compact, there exists a convergent subsequence $\{x_{n_k}\}_{k\geq 1}$ which converges to a point $x_0 \in A$, i.e.

$$\lim_{k\to\infty} x_{n_k} = x_0. \tag{1.3}$$

Because x_0 belongs to A, we can find an open set $\mathcal{O}_{\alpha(x_0)}$ which contains x_0, thus we can find $\epsilon_1 > 0$ such that

$$B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}. \tag{1.4}$$

Now (1.3) implies that there exists $K > 0$ large enough such that:

$$d(x_{n_k}, x_0) \leq \epsilon_1/4, \quad \text{whenever} \quad k > K. \tag{1.5}$$

If y belongs to $B_{\epsilon_1/4}(x_{n_k})$ (i.e. $d(y, x_{n_k}) < \epsilon_1/4$), then the triangle inequality implies (use also (1.5)):

$$d(y, x_0) \leq d(y, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon_1/2 < \epsilon_1, \quad k > K.$$

But this shows that we must have $y \in B_{\epsilon_1}(x_0)$, or:

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq B_{\epsilon_1}(x_0) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K. \tag{1.6}$$

Thus we got the inclusion

$$B_{\epsilon_1/4}(x_{n_k}) \subseteq \mathcal{O}_{\alpha(x_0)}, \quad \forall k > K,$$

which shows that $\epsilon_1/4$ must be less or equal than $2\epsilon_{x_{n_k}}$, or $\epsilon_1/8 \leq \epsilon_{x_{n_k}}$, for every $k > K$. But this is in contradiction with the fact that $\epsilon_{x_n} \leq 1/n$ for every $n \geq 1$. □

Finishing the proof of Theorem 1.8. We now use Proposition 1.7, and find a finite ϵ_0-net for A. Thus we can choose $\{y_1, ...y_N\} \subseteq A$ such that

$$A \subseteq \bigcup_{n=1}^N B_{\epsilon_0}(y_n) \subseteq \bigcup_{n=1}^N B_{\epsilon_{y_n}}(y_n) \subseteq \bigcup_{n=1}^N \mathcal{O}_n,$$

where $\mathcal{O}_n$ is one of the possibly many other open sets which contain $B_{\epsilon_{y_n}}(y_n)$. We have thus extracted our finite subcovering of A and the proof of the theorem is over. □

2 Continuous functions on compact sets

Proposition 2.1. *Let (X, d) be a metric space, $(Y, ||\cdot||)$ a normed space, and H a non-empty, compact subset of X. We define*

$$C(H; Y) := \{f : H \to Y \mid f \text{ is continuous}\}.$$

We also define the map:

$$||\cdot||_\infty : C(H; Y) \to \mathbb{R}_+, \quad ||f||_\infty := \sup_{x\in H} ||f(x)||.$$

Then $(C(H; Y), ||\cdot||_\infty)$ is a normed space.

Proof. We start by showing that $||f||_\infty < \infty$ for every continuous f.

First, due to the inequality $|\ ||y|| - ||y_0||\ | \leq ||y - y_0||$ for every $y, y_0 \in Y$, we easily get that the map $Y \ni y \to ||y|| \in \mathbb{R}_+$ is continuous. Second, for every $f \in C(H; Y)$, the map

$$H \ni x \to ||f(x)|| \in \mathbb{R}_+$$

is a continuous real valued function, defined on a compact set. Then Theorem 10.63 in Wade says that we can find $x_M \in H$ such that

$$\sup_{x\in H} ||f(x)|| = ||f(x_M)|| < \infty.$$

Finally, let us prove the triangle inequality. Take $f, g \in C(H; Y)$; then for every $x \in H$ we apply the triangle inequality in $(Y, ||\cdot||)$:

$$||f(x) + g(x)|| \leq ||f(x)|| + ||g(x)|| \leq ||f||_\infty + ||g||_\infty.$$

Thus $||f||_\infty + ||g||_\infty$ is an upper bound for the set $\{||f(x) + g(x)|| : \ x \in H\}$, hence

$$\sup_{x\in H} ||f(x) + g(x)|| = ||f + g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

□

Proposition 2.2. *Assume that $(Y, ||\cdot||)$ is a Banach space. Then the normed space $(C(H; Y), ||\cdot||_\infty)$ is a Banach space, too.*

Proof. We need to prove that every Cauchy sequence is convergent. Assume that $\{f_n\}_{n\geq 1} \subset C(H; Y)$ is Cauchy, i.e. for every $\epsilon > 0$ one can find $N_C(\epsilon) > 0$ such that $||f_p - f_q||_\infty < \epsilon$ if $p, q > N_C(\epsilon)$. We have to show that the sequence has a limit f which belongs to $C(H; Y)$.

We first construct f. For every $x_0 \in H$ we consider the sequence $\{f_n(x_0)\}_{n\geq 1} \subset Y$. Note the difference between $\{f_n(x_0)\}_{n\geq 1}$ (a sequence of vectors from Y) and $\{f_n\}_{n\geq 1}$ (a sequence of functions from $C(H; Y)$). It is easy to see that $\{f_n(x_0)\}_{n\geq 1}$ is Cauchy in Y (exercise), and because Y is complete, then $\{f_n(x_0)\}_{n\geq 1}$ has a limit in Y. We denote it with $f(x_0)$.

Second, we prove the "uniform convergence" part, or the convergence in the norm $||\cdot||_\infty$. More precisely, it means that for every $\epsilon > 0$ we must construct $N_1(\epsilon) > 0$ so that:

$$\sup_{x\in H} ||f(x) - f_n(x)|| < \epsilon \quad \text{whenever} \quad n > N_1(\epsilon). \tag{2.1}$$

In order to do that, take an arbitrary point $x \in H$. For every $p, n \geq 1$ we have

$$\begin{aligned} ||f(x) - f_n(x)|| &\leq ||f(x) - f_p(x)|| + ||f_p(x) - f_n(x)|| \\ &\leq ||f(x) - f_p(x)|| + ||f_p - f_n||_\infty. \end{aligned} \tag{2.2}$$

If we choose $n, p > N_C(\epsilon/2)$, then we have $||f_p - f_n||_\infty < \epsilon/2$ and

$$||f(x) - f_n(x)|| \leq ||f(x) - f_p(x)|| + \epsilon/2, \quad n, p > N_C(\epsilon/2).$$

But the above left hand side does not depend on p, thus if we take $p \to \infty$ on the right hand side, we get:

$$||f(x) - f_n(x)|| \leq \epsilon/2 < \epsilon, \quad n > N_C(\epsilon/2). \tag{2.3}$$

Note that this inequality holds true *for every* x. This means that $\epsilon/2$ is an upper bound for the set $\{||f(x) - f_n(x)|| : x \in H\}$, hence (2.1) holds true with $N_1(\epsilon) = N_C(\epsilon/2)$.

Third, we must prove that f is a continuous function on H. Fix some point $a \in H$. Choose $\epsilon > 0$. Since $\lim_{n\to\infty} f_n(a) = f(a)$, we can find $N_2(\epsilon, a) > 0$ such that $||f_n(a) - f(a)|| < \epsilon/3$ whenever $n > N_2$. We define $n_1 := \max\{N_1(\epsilon/3) + 1, N_C(\epsilon/3) + 1, N_2 + 1\}$. Because f_{n_1} is continuous at a, we can find $\delta(\epsilon, a) > 0$ so that for every $x \in H$ with $d(x, a) < \delta$ we have $||f_{n_1}(x) - f_{n_1}(a)|| < \epsilon/3$. Thus

$$\begin{aligned} ||f(x) - f(a)|| &\leq ||f(x) - f_{n_1}(x)|| + ||f_{n_1}(x) - f_{n_1}(a)|| + ||f_{n_1}(a) - f(a)|| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned} \tag{2.4}$$

We used (2.1) in order to replace the first and the third term with $\epsilon/3$, and continuity of f_{n_1} at a for the second term. Since a is arbitrary, we can conclude that f is continuous on H, thus belongs to $C(H; Y)$. Therefore we can rewrite (2.1) as:

$$||f - f_n||_\infty < \epsilon \quad \text{whenever} \quad n > N_1(\epsilon), \tag{2.5}$$

and the proof is over. □

Remark 2.3. *The "ordinary" convergence in the functional space* $(C(H; Y), d_\infty)$ *(given in (2.5)) is nothing but the uniform convergence of a sequence of functions defined on the set* H *(see (2.1)). One can find more details in Wade, exercise 6 in Chapter 10.6 (page 314).*

3 Compact sets in $(C(H; \mathbb{R}^n), || \cdot ||_\infty)$

We here are interested in finding some sufficient conditions for a subset of $(C(H; \mathbb{R}^n), || \cdot ||_\infty)$, $n \geq 1$, in order to be compact. (We know that in the Euclidian space $(\mathbb{R}^n, || \cdot ||)$ a set is compact if and only if it is bounded and closed; this is the Heine-Borel theorem).

Definition 3.1. *We say that* $f : H \to Y$ *is uniformly continuous if for every* $\epsilon > 0$*, we can find* $\delta(f, \epsilon) > 0$ *such that for all points* $x, y \in H$ *which fulfill* $d(x, y) \leq \delta(f, \epsilon)$ *we have that* $||f(x) - f(y)|| \leq \epsilon$*.*

Theorem 9.32 in Wade (Heine's theorem) shows that a function $f : H \to \mathbb{R}^n$ is continuous if and only if it is uniformly continuous.

Definition 3.2. *A family of functions $K \subset C(H; \mathbb{R}^n)$ is called* equibounded *if there exists a constant $M_K < \infty$ such that*

$$\sup_{x \in H} ||f(x)|| = ||f||_\infty \leq M_K, \quad \forall f \in K. \tag{3.1}$$

Definition 3.3. *A family of functions $K \subset C(H; \mathbb{R}^n)$ is called* uniformly equicontinuous *if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, such that for every $f \in K$ and for every pair of points $x, y \in H$ which obey $d(x, y) \leq \delta(\epsilon)$, one has that $||f(x) - f(y)|| \leq \epsilon$. In other words, (see Definition 3.1)*

$$\inf_{f \in K} \delta(f, \epsilon) = \delta_K(\epsilon) > 0. \tag{3.2}$$

Definition 3.4. *A subset Z of a metric space (M, d) is called dense in M if every point $x \in M$ is the limit of a sequence $\{x_n\}_{n \geq 1} \subseteq Z$. A set Z is called countable if there exists a map $j : Z \to \mathbb{N}$ which is injective. A metric space is called separable if it contains a countable dense subset.*

Theorem 3.5. *(*Arzela-Ascoli*). Let (X, d) be a metric space, and let H be a compact subset of X. Assume that $Z \subset H$ is countable and dense in H (i.e. (H, d) is separable). Denote by $K \subset C(H; \mathbb{R}^n)$ the family of* all *functions which are equibounded by some M_K and uniformly equicontinuous with some δ_K (ækvibegrænset og uniformt ækvikontinuert). Then K is sequentially compact (følgekompakt) and thus compact. The closure in $C(H; \mathbb{R}^n)$ of any subset of K is also compact.*

Proof. We will show that given an arbitrary sequence of functions $\{f_n\}_{n \geq 1} \subset K$, one can always find a subsequence which converges to a "point" in K (note that a point in K means a function defined on H; we denote this "point" with f). This would prove that K is sequentially compact.

Because the dense set Z is countable, we can represent it in the following way:

$$Z = \{z_1, z_2, z_3, ...\}.$$

The sequence $\{f_n(z_1)\}_{n \geq 1} \subset \mathbb{R}^n$ is bounded because we have $||f_n(z_1)|| \leq M_K$ for every n, see (3.1). The Bolzano-Weierstrass theorem allows us to find a subsequence $\{f_{n_1}(z_1)\}_{n_1 \geq 1} \subset \mathbb{R}^n$, which converges to a point in $\mathbb{R}^n$; we call this point with $f(z_1)$.

Now consider the sequence $\{f_{n_1}(z_2)\}_{n_1 \geq 1} \subset \mathbb{R}^n$. This sequence is also bounded, thus we can find a second subsequence

$$\{f_{n_2}(z_2)\}_{n_2 \geq 1} \subseteq \{f_{n_1}(z_2)\}_{n_1 \geq 1},$$

which converges to a point in $\mathbb{R}^n$; we call this point with $f(z_2)$. Note that the subsequence of *functions* $\{f_{n_2}\}_{n_2 \geq 1} \subseteq \{f_{n_1}\}_{n_1 \geq 1}$ converges pointwise in both z_1 and z_2.

We can continue this procedure and obtain a subsequence of functions $\{f_{n_p}\}_{n_p \geq 1}$ where

$$\{f_{n_p}\}_{n_p \geq 1} \subseteq \{f_{n_{p-1}}\}_{n_{p-1} \geq 1} \subseteq ... \subseteq \{f_n\}_{n \geq 1}$$

and $\{f_{n_p}\}_{n_p \geq 1}$ converges pointwise in all the points $\{z_1, ..., z_p\}$ towards the limits $\{f(z_1), ..., f(z_p)\}$. More precisely, for every $\epsilon > 0$, there exists $N(p, \epsilon) > 1$ such that

$$||f_{n_p}(z_k) - f(z_k)|| < \epsilon, \quad \text{whenever} \quad n_p > N(p, \epsilon), \ k \in \{1, ..., p\}. \tag{3.3}$$

Introduce the notation $N_p := N(p, 1/p) + 1$. Then we have the important estimate:

$$||f_{N_p}(z_k) - f(z_k)|| < 1/p, \quad \text{whenever} \quad k \in \{1, ..., p\}. \tag{3.4}$$

This means that we have constructed a "diagonal subsequence" $\{f_{N_p}\}_{p\geq 1}$ having the property that $\{f_{N_p}(z_k)\}_{p\geq 1} \subset \mathbb{R}^n$ is convergent for every fixed k, and we denote the limits with:

$$\lim_{p\to\infty} f_{N_p}(z_k) = f(z_k), \quad k \text{ fixed.} \tag{3.5}$$

This is the same thing as to say that the sequence $\{f_{N_p}\}_{p\geq 1}$ converges pointwise on Z:

$$\lim_{p\to\infty} f_{N_p}(z) = f(z), \quad \forall z \in Z. \tag{3.6}$$

In the next lemma we will show that the sequence $\{f_{N_p}\}_{p\geq 1}$ is a Cauchy sequence in $C(H; \mathbb{R}^n)$. Let us now assume that this holds true, and let us prove the Arzela-Ascoli theorem.

If this sequence is Cauchy, then according to Proposition 2.2 it will have a limit in $C(H; \mathbb{R}^n)$, which we denote by F. But then F is continuous on H and equal to $f(z)$ for every $z \in Z$. The only thing remained to prove is that $F \in K$, i.e. to verify that F verifies (3.1) and (3.2).

First, (3.1) follows from:

$$||F(x)|| = \lim_{p\to\infty} ||f_{N_p}(x)||, \quad ||f_{N_p}(x)|| \leq M_K, \quad x \in H,$$

and (3.2) from:

$$\begin{aligned} ||F(x) - F(y)|| &= \lim_{p\to\infty} ||f_{N_p}(x) - f_{N_p}(y)||, \\ ||f_{N_p}(x) - f_{N_p}(y)|| &\leq \epsilon \quad \text{whenever} \quad d(x,y) \leq \delta(\epsilon). \end{aligned} \tag{3.7}$$

Thus $F \in K$, and the theorem is proved. Hence the only remaining technical ingredient is the following lemma:

Lemma 3.6. *For every $\epsilon' > 0$, there exists $N_C(\epsilon') > 0$ such that for every $p, q > N_C(\epsilon')$ we have*

$$\sup_{x\in H} ||f_{N_p}(x) - f_{N_q}(x)|| = ||f_{N_p} - f_{N_q}||_\infty < \epsilon'.$$

Bevis. Choose $0 < \epsilon < \epsilon'$. Consider $\delta_K(\epsilon/3)$ as it was defined in (3.2).

Let us now show that

$$\{B_{\delta_K(\epsilon/3)/2}(z_j) : \; z_j \in H\}$$

is an open covering of H. First, because Z is dense in H, then for every point $x \in H$ there exists a sequence $\{x_m\}_{m\geq 1} \subset Z$ such that $x_m \to x$. Second, we may find $x_M \in Z$ such that $B_{\delta_K(\epsilon/3)/3}(x) \subset B_{\delta_K(\epsilon/3)/2}(x_M)$ provided $d(x, x_M) < \delta_K(\epsilon/3)/6$ (exercise). Thus we can write:

$$H \subset \bigcup_{x\in H} B_{\delta_K(\epsilon/3)/3}(x) \subset \bigcup_{k=1}^{\infty} B_{\delta_K(\epsilon/3)/2}(z_k).$$

Because H is compact, we can extract a finite open subcovering:

$$H \subseteq \bigcup_{l=1}^{m(\epsilon)} B_{\delta_K(\epsilon/3)/2}(z_{k_l}). \tag{3.8}$$

Now take an arbitrary point $x \in H$. We can find some $l \in \{1, ..., m(\epsilon)\}$ such that $x \in B_{\delta_K(\epsilon/3)/2}(z_{k_l})$. We can write:

$$\begin{aligned} & ||f_{N_p}(x) - f_{N_q}(x)|| && (3.9)\\ \leq\ & ||f_{N_p}(x) - f_{N_p}(z_{k_l})|| + ||f_{N_p}(z_{k_l}) - f_{N_q}(z_{k_l})|| + ||f_{N_q}(z_{k_l}) - f_{N_q}(x)||. \end{aligned}$$

Because K is uniformly equicontinuous, and because $d(x, z_{k_l}) < \delta_K(\epsilon/3)$, then the first and third term in the right hand side of (3.9) are less than $\epsilon/3$ (see Definition 3.3), uniformly in p and q. Thus

$$||f_{N_p}(x) - f_{N_q}(x)|| \leq 2\epsilon/3 + ||f_{N_p}(z_{k_l}) - f_{N_q}(z_{k_l})||, \quad \forall p, q \geq 1. \tag{3.10}$$

Note the very important thing that there only are a finite number of points of the type z_{k_l}, i.e. $m(\epsilon)$ of them. Hence (3.5) implies that the $m(\epsilon)$ sequences $\{f_{N_r}(z_{k_l})\}_{r\geq 1} \subset \mathbb{R}^n$ are all Cauchy at the same time; we can thus find a large enough index $N_1(\epsilon/3)$ such that if $p, q > N_1(\epsilon/3)$ then

$$||f_{N_p}(z_{k_l}) - f_{N_q}(z_{k_l})|| < \epsilon/3, \quad 1 \leq l \leq m(\epsilon).$$

Use this in (3.10) and obtain:

$$||f_{N_p}(x) - f_{N_q}(x)|| < \epsilon, \quad \text{whenever} \quad p, q \geq N_C(\epsilon') := N_1(\epsilon/3). \tag{3.11}$$

Because $N_C(\epsilon')$ is independent of x, we can write

$$\sup_{x\in H} ||f_{N_p}(x) - f_{N_q}(x)|| \leq \epsilon < \epsilon', \quad p, q \geq N_C(\epsilon')$$

and the lemma is proved, and so is the theorem. □

4 The completion of a normed space

Theorem 4.1. *Let $(A, || \cdot ||_a)$ be a normed space. Then there exists a Banach space B with a norm $|| \cdot ||_b$ and a linear mapping $I : A \to B$ such that*

$$||I(x)||_b = ||x||_a, \quad \forall x \in A,$$

and $I(A)$ is dense in B with respect to $|| \cdot ||_b$. We call $(B, || \cdot ||_b)$ the completion of $(A, || \cdot ||_a)$.

Proof. Let us denote by

$$l^\infty(A) := \{\psi : \mathbb{N} \to A : \sup_n ||\psi(n)||_a < \infty\}.$$

This is nothing but the space of all bounded sequences with elements in A. We now organize $l^\infty(A)$ as a linear space. If ψ and ϕ are in $l^\infty(A)$, then $\alpha\psi + \beta\phi$

is also a bounded sequence, for every α and β in the scalar field of A. A norm on $l^\infty(A)$ is:

$$||\psi||_\infty := \sup_{n\geq 1} ||\psi(n)||_a. \tag{4.12}$$

We consider the linear subspace

$$c_0(A) := \{\psi \in l^\infty(A) : \lim_{n\to\infty} ||\psi(n)||_a = 0\}.$$

Since $c_0(A) \subset l^\infty(A)$, we can define the quotient (the space of all equivalence classes of $l^\infty(A)$ with respect to $c_0(A)$) in the following way:

$$\begin{aligned} &A \ni \psi \mapsto \Psi := \{\psi + f : \ \forall f \in c_0(A)\} \\ &Y := \{\Psi : \ \forall \psi \in A\}. \end{aligned} \tag{4.13}$$

Thus Y is a set of sets containing functions. We call ψ a representative of Ψ; any other function of the type $\psi + f$ where $f \in c_0(A)$ is a representative for Ψ.

We organize Y as a linear space in the usual way. For any two scalars α, β, and for any two equivalence classes Ψ, Φ (with representatives ψ, ϕ), we define $\alpha\Psi + \beta\Phi$ to be the equivalence class associated to $\alpha\psi + \beta\phi$.

Let us now define an application on Y, which we later on will prove it is a norm:

$$Y \ni \Psi \to ||\Psi||_y := \inf_{g\in c_0(A)} ||\psi + g||_\infty \in \mathbb{R}_+. \tag{4.14}$$

It is easy to see that the application is well-defined (i.e. it is independent of the choice we make for the representative ψ; prove it!).

We have a first result:

Lemma 4.2. *Let $\psi \in \Psi$. Then*

$$||\Psi||_y = \limsup_{n\to\infty} ||\psi(n)||_a = \inf_{n\geq 1}\{\sup_{m\geq n} ||\psi(m)||\}. \tag{4.15}$$

Proof. We first prove:

$$||\Psi||_y \leq \inf_{n\geq 1}\{\sup_{m\geq n} ||\psi(m)||\}. \tag{4.16}$$

In order to do that, define the function $f_n \in c_0(A)$ such that $f_n(m) = -\psi(m)$ for $m < n$ and $f_n(m) = 0$ if $m \geq n$. Then the function $\psi + f_n$ is different from zero only for $m > n$ and we have (see definition (4.12))

$$||\psi + f_n||_\infty = \sup_{m\geq n} ||\psi(m)||_a.$$

Then since $||\Psi||_y$ is the infimum over functions in $c_0(A)$, we can write

$$||\Psi||_y \leq ||\psi + f_n||_\infty = \sup_{m\geq n} ||\psi(m)||_a.$$

Thus $||\Psi||_y$ is a lower bound for the sequence $\{\sup_{m\geq n} ||\psi(m)||_a\}_{n\geq 1}$, and so it is smaller than the largest lower bound of the sequence, hence (4.16) holds true.

Now we prove the reversed inequality:

$$\inf_{n\geq 1}\{\sup_{m\geq n} ||\psi(m)||\} \leq ||\Psi||_y. \tag{4.17}$$

Choose $\epsilon > 0$. Since $||\Psi||_y$ is an infimum given in (4.14), then there exists $g_\epsilon \in c_0(A)$ such that

$$||\psi + g_\epsilon||_\infty \leq ||\Psi||_y + \epsilon/2. \tag{4.18}$$

From the definition of the infinity norm, we see that

$$\sup_{m\geq n} ||\psi(m) + g_\epsilon(m)||_a \leq ||\psi + g_\epsilon||_\infty, \quad \forall n \geq 1, \tag{4.19}$$

because we take the supremum over a smaller set of indices. The triangle inequality gives us

$$||\psi(m)||_a - ||g_\epsilon(m)||_a \leq ||\psi(m) + g_\epsilon(m)||_a, \quad \forall m \geq n. \tag{4.20}$$

Because g_ϵ is in $c_0(A)$, it obeys $\lim_{n\to\infty} ||g_\epsilon(n)||_a = 0$. Thus we can find n_ϵ such that for every $m \geq n_\epsilon$ we have

$$||g_\epsilon(m)||_a \leq \epsilon/2, \quad \forall m \geq n_\epsilon. \tag{4.21}$$

Use (4.21) in (4.20) with $n = n_\epsilon$. We obtain:

$$||\psi(m)||_a - \epsilon/2 \leq ||\psi(m) + g_\epsilon(m)||_a, \quad \forall m \geq n_\epsilon. \tag{4.22}$$

Take the supremum over $m \geq n_\epsilon$ and use (4.19):

$$\sup_{m\geq n_\epsilon} ||\psi(m)||_a - \epsilon/2 \leq ||\psi + g_\epsilon||_\infty. \tag{4.23}$$

Use (4.18) and write:

$$\inf_{n\geq 1}\{\sup_{m\geq n} ||\psi(m)||\} \leq \sup_{m\geq n_\epsilon} ||\psi(m)||_a \leq ||\Psi||_y + \epsilon, \quad \forall \epsilon > 0,$$

and since ϵ is arbitrary, the proof of the lemma is over. □

The next result states that Y is a normed space.

Lemma 4.3. *The map $||\cdot||_y$ is a norm.*

Proof. (i). If $\Psi = c_0(A)$ (i.e. the "zero" equivalence class, its representative being the zero function), then clearly $||\Psi||_y = 0$. Now assume $||\Psi||_y = 0$. Pick any representative ψ. Use (4.15) and get $\limsup_{n\to\infty} ||\psi(n)||_a = 0$. But this implies that $\lim_{n\to\infty} ||\psi(n)||_a = 0$ hence $\psi \in c_0(A)$ and $\Psi = c_0(A)$.

(ii). Use (4.15) to prove the homogeneity: $||\lambda\Psi||_y = |\lambda| \cdot ||\Psi||_y$.

(iii). For the triangle inequality, consider Ψ and Φ in Y, with representatives ψ and ϕ. Choose $\epsilon > 0$. There exist $f_\epsilon, g_\epsilon \in c_0(A)$ such that $||\psi + f_\epsilon||_\infty \leq ||\Psi||_y + \epsilon/2$ and $||\phi + g_\epsilon||_\infty \leq ||\Phi||_y + \epsilon/2$. But since $f_\epsilon + g_\epsilon \in c_0(A)$, we also have that

$$||\Psi + \Phi||_y \leq ||\psi + \phi + f_\epsilon + g_\epsilon||_\infty \leq ||\psi + f_\epsilon||_\infty + ||\phi + g_\epsilon||_\infty \leq ||\Psi||_y + ||\Phi||_y + \epsilon$$

where we applied the triangle inequality for the $||\cdot||_\infty$ norm, and we are done. □

The last preparatory result is the following:

Lemma 4.4. *The normed space* $(Y, ||\cdot||_y)$ *is a Banach space.*

Proof. We only need to show that every Cauchy sequence with elements in Y has a limit in Y. Consider a Cauchy sequence $\{\Psi_p\}_{p\geq 1} \subset Y$. It means that for every $\epsilon > 0$, we can find p_ϵ large enough such that

$$||\Psi_{k+p} - \Psi_p||_y < \epsilon, \quad \forall k \geq 0,\ \forall p \geq p_\epsilon. \tag{4.24}$$

If we choose $\epsilon = 1/2^j$ for every $j \geq 1$, we obtain a sequence of indices p_j given by the above p_ϵ's. We can assume without loss of generality that p_j is a strictly increasing sequence with j. Now define

$$\Phi_j := \Psi_{p_j}, \quad j \geq 1.$$

Note that (4.24) implies:

$$||\Phi_{k+j} - \Phi_j||_y < 2^{-j}, \quad \forall k \geq 1,\ \forall j \geq 1. \tag{4.25}$$

The strategy is to construct a limit for the subsequence $\{\Phi_j\}_{j\geq 1}$, which (as is well-known) will also be a limit for the whole sequence.

A representative of Φ_j is denoted by ϕ_j. Using (4.25) and (4.15), we have

$$\inf_{n\geq 1}\{\sup_{m\geq n} ||\phi_{k+j}(m) - \phi_j(m)||_a\} < 2^{-j}$$

which means that 2^{-j} is not a lower bound for the n-depending sequence $\{\sup_{m\geq n} ||\phi_{k+j}(m) - \phi_j(m)||_a\}_{n\geq 1}$. This leads to the observation that we can find $n(j)$ large enough such that

$$||\phi_{j+1}(m) - \phi_j(m)||_a < 2^{-j}, \quad \forall m \geq n(j). \tag{4.26}$$

Without loss of generality, we can choose $n(j)$ to be strictly increasing with j.

Now define a particular Ψ_∞ having the representative given by $\psi_\infty(m) = \phi_{j+1}(m)$ if $n(j) \leq m < n(j+1)$. For a fixed $p \geq 1$ we have

$$||\Psi_\infty - \Phi_p||_y \leq \sup_{m\geq n} ||\psi_\infty(m) - \phi_p(m)||_a, \quad \forall n \geq 1.$$

Now choose $n = n(p)$ in the above inequality, and let us estimate the supremum on the right hand side. We can exhaust the interval $[n(p), \infty)$ by considering all situations in which $m \in [n(j), n(j+1))$, and $j \geq p$. For such an m we can write:

$$\begin{aligned} ||\psi_\infty(m) - \phi_p(m)||_a &= ||\sum_{k=p}^{j}(\phi_{k+1}(m) - \phi_k(m)||_a \\ &\leq \sum_{k=p}^{j} ||\phi_{k+1}(m) - \phi_k(m)||_a. \end{aligned} \tag{4.27}$$

Because $m \geq n(j) > n(j-1) > ... > n(p)$ it follows (see (4.26)):

$$||\phi_{k+1}(m) - \phi_k(m)||_a < 2^{-k}, \quad k \in \{p, p+1, ..., j\}. \tag{4.28}$$

Using this in (4.27) we obtain that for every $m \in [n(j), n(j+1))$ we have the inequality

$$||\psi_\infty(m) - \phi_p(m)||_a \leq \sum_{k=p}^{j} 2^{-k} \leq \sum_{k=p}^{\infty} 2^{-k} = 2^{-p+1}$$

and this estimate in now independent of j, thus the supremum over $m \geq n(p)$ cannot be larger than 2^{-p+1}. We therefore proved that $||\Psi_\infty - \Phi_p||_y \leq 2^{-p+1}$ for all $p \geq 1$, thus the subsequence $\{\Phi_p\}_{p\geq 1}$ is convergent, hence the whole original sequence $\{\Psi_n\}_{n\geq 1}$ converges to Ψ_∞. The proof is over. $\square$

End of the proof of Theorem 4.1. Now we can construct the Banach space B. For every $x \in A$, define the function $i(x) \in l^\infty(A)$ which obeys $[i(x)](n) = x$ for every $n \geq 1$. Denote by $I(x)$ the equivalence class in Y whose representative is $i(x)$. Since $||[i(x)](m)||_a = ||x||_a$ for every $m \geq 1$, then using (4.15) we obtain $||I(x)||_y = ||x||_a$.

Now consider the subset of Y defined by

$$I(A) := \{I(x) : \; \forall x \in A\}.$$

It clearly is a linear subspace of Y (exercise), which can be organized as a normed space with the norm induced by $|| \cdot ||_y$. Redenote the old norm $|| \cdot ||_y$ by $|| \cdot ||_b$. Then by taking the closure of $I(A)$ in Y with respect to $|| \cdot ||_b$ we obtain a closed linear subspace $\overline{I(A)}$ which is also a Banach space since Y is a Banach space (exercise). Therefore $B = \overline{I(A)}$ and we conclude the theorem. $\square$

5 Zorn's Lemma

A set S is partially ordered if there exists an order relation $\leq$ which is reflexive ($x \leq x$ for all x), antisymmetric (if $x \leq y$ and $y \leq x$ then $x = y$) and transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$). If $x \leq y$ and $x \neq y$, then we write $x < y$ or $y > x$.

A chain in S is a subset C in which any two elements are comparable, that is for every $x, y \in C$ then either $x \leq y$ or $y \leq x$.

An element $m \in S$ is called maximal if there is no other $x \in S$ such that $m < x$. This does not mean that m is the largest element, which would be an element $M \in S$ such that $x \leq M$ for every $x \in S$.

Theorem 5.1. *(Zorn's lemma). Let S be a partially ordered set in which every chain has an upper bound. Then S has at least one maximal element.*

Proof. We first prove a weaker version, in which we assume that every chain has a least upper bound (supremum). More precisely, for every chain C there exists an element called $\sup(C)$ which *(i)* is an upper bound for C, i.e. for every $x \in C$ we have $x \leq \sup(C)$, and *(ii)* is the smallest upper bound, i.e. for every $x \in C$ such that $x < \sup(C)$ there exists $z \in C$ such that $x < z \leq \sup(C)$.

Proposition 5.2. *Let S be a partially ordered set in which every chain has a supremum. Then S has at least one maximal element.*

Proof. Define a "successor" operation on S as follows: if x is non-maximal, choose some $y > x$ and set $\phi(x) = y$. If x is maximal, put $\phi(x) = x$. Note that the existence of ϕ is insured by the axiom of choice.

Now we say that a subset $N \subseteq S$ is a *tower* if we have the following two properties:

$$P1. \text{ If } x \in N, \text{ then } \phi(x) \in N;$$
$$P2. \text{ For any chain } C \subseteq N, \text{ then } \sup(C) \in N.$$

Let us note that S itself is a tower, and the intersection of any family of towers is a tower (exercise). In particular, the intersection of all possible towers is a tower. Denote the smallest (non-empty) tower of S with M.

Definition 5.3. *($P3$) We say that $x \in M$ has the property $P3$ if for any $y \in M$, we either have $y \leq x$ or $y \geq \phi(x)$.*

Lemma 5.4. *Assume that $P3$ holds for all $x \in M$. Then M is a chain, and M has a largest element which also is a maximal element of S.*

Proof. Let us prove that M is chain. For, take $x, y \in M$. Then $P3$ applied for x says that if we do not have $y \leq x$, then we must have $\phi(x) \leq y$. But $\phi(x) \in M$ due to $P1$, and $x \leq \phi(x)$. The transitivity then gives $x \leq y$, hence M is a chain. Now because M is a chain, then due to $P2$ it must contain its supremum $\sup(M)$. But then $\sup(M)$ is a maximal element, because on one hand $\phi(\sup(M)) \geq \sup(M)$, and on the other hand due to $P1$ we have that $\phi(\sup(M)) \in M$ hence $\phi(\sup(M)) \leq \sup(M)$. The antisymmetry gives $\phi(\sup(M)) = \sup(M)$ and we get our maximal element, thus proving the lemma. $\square$

From the above lemma we see that the proposition is proved if we can show that $P3$ holds for all point of M. In order to do that, we first need another definition:

Definition 5.5. *($P4$) We say that $x \in M$ has property $P4$ if for any $y \in M$ with $y < x$ we have $\phi(y) \leq x$.*

Lemma 5.6. *If $x \in M$ obeys $P4$, then it also obeys $P3$.*

Proof. Let
$$M' := \{y \in M : \; y \leq x \quad \text{or} \quad y \geq \phi(x)\}.$$
If we can prove that M' is a tower, then $M' = M$ because M is the smallest tower of S. So we need to verify $P1$ and $P2$ for M'. We start with $P1$, that is we need to show that for any $y \in M'$ we have $\phi(y) \in M'$. Indeed, if $y \in M'$ then we either have a) $y < x$, b) $y = x$ or c) $y \geq \phi(x)$. If a) holds, then $P4(x)$ says that $\phi(y) \leq x$ hence $\phi(y) \in M'$, thus $P1$ holds. If either b) or c) holds, then we trivially have $\phi(y) \geq \phi(x)$, thus $\phi(y) \in M'$, hence $P1$ holds.

In order to verify $P2$, we need to show that if $C \subseteq M'$ is a chain, then $\sup(C) \in M'$. Clearly, because C is also a chain in M, we have that $\sup(C) \in M$.

Now we have two possibilities: a) $z \leq x$ for all $z \in C$, and b) there exists some $z \in C$, $z > x$. If a) holds, then x is an upper bound hence $\sup(C) \leq x$, thus $\sup(C) \in M'$. If b) holds, then because $z > x$ and $z \in M'$ implies that $z \geq \phi(x)$, hence $\sup(C) \geq \phi(x)$, thus $\sup(C) \in M'$. Since $P2$ is also verified, then M' is a tower and $M = M'$. $\square$

The last step in the proof of the proposition, is showing that $P4$ holds true for every $x \in M$. For, denote by N the set of points $x \in M$ which obey $P4$. As above, it suffices to show that N is a tower.

We start with proving $P1$ for N. Take $x \in N$, and we want to show that $\phi(x) \in N$. For that, look at all $y \in M$ with $y < \phi(x)$ and try to show that $\phi(y) \leq \phi(x)$.

Because we know from Lemma 5.6 that x obeys $P3$, the only possibility is to have $y \leq x$. Then a) $y < x$ or b) $y = x$. If a) holds, then because x was supposed to obey $P4$ we get $\phi(y) \leq x$, hence $\phi(y) \leq \phi(x)$. If b) holds, then trivially $\phi(y) \leq \phi(x)$. In both cases we proved that $\phi(x) \in N$ hence $P1$ is fulfilled.

We now prove that $P2$ holds. Consider a chain $C \subseteq N$; we want to show that $\sup(C) \in N$, i.e. $\sup(C)$ has the property $P4$. In other words, for every $y \in M$ with $y < \sup(C)$ we need to show that $\phi(y) \leq \sup(C)$. Now from $y < \sup(C)$ it means that y is not an upper bound for C, so it exists $z \in C$ such that $z \not\leq y$. This means that either a) y and z are not comparable, or b) $y < z$. But $z \in N$ has property $P4$ and hence $P3$ (from Lemma 5.6), thus z and $y \in M$ are comparable, hence b) holds. Now apply $P4(z)$: it gives $\phi(y) \leq z$, hence $\phi(y) \leq \sup(C)$. Therefore $\sup(C) \in N$, and $P2$ is verified. We conclude that N is a tower, therefore $N = M$.

Finishing the proof of Proposition 5.2. We have just shown that all points of M have the property $P4$. Lemma 5.6 showed that $P4$ implies $P3$. Then Lemma 5.4 says that M must have a largest element, which was shown to be a maximal element of S. Thus Proposition 5.2 is proved. □

We now use Proposition 5.2 for proving the *Hausdorff maximal principle*:

Lemma 5.7. *(The Hausdorff maximal principle). Let Q be a partially ordered set. Then Q contains a maximal chain (i.e. a chain which is not contained in a bigger chain).*

Proof. Define S to be the set of all chains of Q, partially ordered with respect to the set inclusion. More precisely, if C_1 and C_2 are chains in Q (and elements of S), then we say that $C_1 \leq_S C_2$ in S if $C_1 \subseteq C_2$ in Q. It is easy to prove that $\leq_S$ is a partial order (exercise).

Another important property is that the intersection in Q of two chains is a chain, and in fact an arbitrary intersection of chains from Q is a chain (exercise).

Now let us denote an arbitrary chain in S by K. Note that the elements of K in S consist of chains in Q. Denote by $\#K$ the set K seen as a set formed of elements of Q; clearly, $\#K$ is a chain in Q. If Q has no maximal chain, then there should exist at least one element $k \in Q$ such that $x <_Q k$ for all $x \in \#K$. Then $\#\tilde{K} := \#K \cup \{k\}$ is a chain in Q, and $\tilde{K} := K \cup \{k\}$ is an upper bound for K in S.

Now if A_1 and A_2 in S are upper bounds for K, then $A_1 \cap A_2$ is also an upper bound for K (exercise). Define $\sup(K)$ as the intersection of all possible upper bounds of K; then $\sup(K)$ is a chain in Q, and by construction it is a least upper bound for K.

Therefore S is a partially ordered set where all chains K have a supremum. Proposition 5.2 now states that there exists a maximal element $K \in S$. But this is the same with saying that $\#K$ is a chain in Q which is not included in a longer chain, and the proof of this lemma is over.

Finishing the proof of Zorn's Lemma. We can now lift the extra-condition in Proposition 5.2. Assume that S is a partially ordered set, where every chain has an upper bound. According to the Hausdorff maximum principle, there exists a maximal chain $C \subseteq S$. Being a chain, C must have an upper bound $x \in S$, and this means that $C \cup \{x\}$ is another chain in S. But C is maximal, therefore $x \in C$. Moreover, x must be the largest element of C. Finally, x is a maximal element in S and $\phi(x) = x$, because if $\phi(x) > x$ we can consider $C \cup \{\phi(x)\}$ which would contradict the maximality of C. The proof of the theorem is over. □

6 Baire's Category Theorem

Denote the open ball of radius ϵ and centred at x by $B_\epsilon(x) := \{y \in B : ||y-x|| < \epsilon\}$. The complementary in B of a set $S \subseteq B$ is denoted by S^c.

Theorem 6.1. *Consider a Banach space B, and a sequence of closed sets $\{S_n\}_{n\geq 1}$ such that*

$$B = \bigcup_{n\geq 1} S_n. \tag{6.29}$$

Then there exists at least one set S_n with non-empty interior.

Proof. Assume the contrary, that is each S_n has an empty interior. One can re-state this in a more formal way: for every $x \in S_n$, and for every $\epsilon > 0$, we have:

$$B_\epsilon(x) \cap S_n^c \neq \emptyset, \quad \forall \epsilon > 0. \tag{6.30}$$

We can assume that all sets S_n are non-empty. We also have that $S_n^c \neq \emptyset$, since otherwise $S_n = B$ which would have a non-empty interior.

Let therefore x_1 be a point of S_1^c. Because S_1 is closed, we have that S_1^c is open, therefore there exists $\epsilon_1 > 0$ such that

$$B_{\epsilon_1}(x_1) \subset S_1^c. \tag{6.31}$$

Starting from x_1 and ϵ_1, we will inductively define two sequences $\{x_n\}_{n\geq 1} \subset B$ and $\{\epsilon_n\}_{n\geq 1} \subset \mathbb{R}_+$, having several properties. First, we need:

$$\epsilon_{n+1} < \frac{\epsilon_n}{3}, \quad n \geq 1. \tag{6.32}$$

Second, we need that:

$$B_{\epsilon_n}(x_n) \subset S_n^c, \quad n \geq 1, \tag{6.33}$$

and third:

$$||x_{n+1} - x_n|| < \frac{\epsilon_n}{3}, \quad n \geq 1. \tag{6.34}$$

Let us investigate the consequences of having such sequences, and we will later on prove their existence. First, (6.32) leads us to the estimate:

$$\epsilon_j < \frac{\epsilon_{j-1}}{3} < ... < \frac{\epsilon_n}{3^{j-n}}, \quad \forall\, j > n \geq 1. \tag{6.35}$$

In particular, $\epsilon_n < \epsilon_1/3^{n-1} \to 0$ when $n \to \infty$.

Second, we can prove that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence, because for every $p \geq 1$ we can write

$$\begin{aligned} ||x_{n+p} - x_n|| &= || \sum_{j=n}^{n+p-1} [x_{j+1} - x_j]|| \leq \sum_{j=n}^{n+p-1} ||x_{j+1} - x_j|| \\ &< \sum_{j=n}^{n+p-1} \epsilon_j/3 < \sum_{j=n}^{\infty} \epsilon_j/3 \\ &< \frac{\epsilon_n}{3} \sum_{k\geq 0} 3^{-k} = \frac{\epsilon_n}{2} \to 0, \quad n \to \infty. \end{aligned} \tag{6.36}$$

In the first line we used the triangle inequality, in the second line we used (6.34), and in the third line (6.35).

Because B is a Banach space, $\{x_n\}_{n\geq 1}$ is convergent and has a limit $x \in B$. But then we have (use the triangle inequality)

$$||x - x_n|| \leq ||x - x_{n+p}|| + ||x_{n+p} - x_n|| < ||x - x_{n+p}|| + \frac{\epsilon_n}{2}, \quad \forall\, p \geq 1.$$

Since $\lim_{p\to\infty} ||x - x_{n+p}|| = 0$, taking p to infinity in the above estimate gives us $||x - x_n|| < \epsilon_n$, or $x \in B_{\epsilon_n}(x_n)$, or $x \in S_n^c$ (see (6.33)), or $x \notin S_n$ for all n. But this contradicts (6.29).

Therefore the only remaining thing is the construction of our sequences. Let us first construct x_2 and ϵ_2.

(i). If $x_1 \in S_2^c$, then put $x_2 = x_1$. Then since S_2^c is open, we can find $\epsilon' > 0$ such that $B_{\epsilon'}(x_1) \subset S_2^c$. Now choose ϵ_2 to be the minimum between ϵ' and $\epsilon_1/4$. Clearly, (6.32) and (6.34) hold true for $n = 1$ (we here have $||x_1 - x_2|| = 0$), while (6.33) holds true for $n = 1, 2$.

(ii). If $x_1 \notin S_2^c$, then of course $x_1 \in S_2$. From (6.30) we have that for every $\epsilon' > 0$ we can find $y(\epsilon') \in B_{\epsilon'}(x_1) \cap S_2^c$, that is $||y(\epsilon') - x_1|| < \epsilon'$. Define $x_2 := y(\epsilon_1/4) \in S_2^c$. Because S_2^c is open, we can find $\epsilon'' > 0$ such that $B_{\epsilon''}(x_2) \subset S_2^c$. Finally define ϵ_2 as the minimum between ϵ'' and $\epsilon_1/4$. Then we have $\epsilon_2 < \epsilon_1/3$, $||x_2 - x_1|| < \epsilon' < \epsilon_1/3$, and $B_{\epsilon_2}(x_2) \subset S_2^c$.

The induction step from x_n and ϵ_n to x_{n+1} and ϵ_{n+1} is identical to the one from 1 to 2. The theorem is proved. $\square$