Schauder’s Fixed Point Theorem

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**Theorem 0.1.** Let $X$ be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f: K \to K$ there exists $x \in K$ such that $f(x) = x$.

Notice that the unit disc of a finite dimensional vector space is always convex and compact hence this theorem extends Brouwer Fixed Point Theorem.

**Proof.** We suppose that $X$ is a Banach space. The idea of the proof is to reduce ourselves to the finite dimensional case. Given $\varepsilon > 0$ notice that the family of open sets $\{B_\varepsilon(x) : x \in K\}$ is an open covering of $K$. Being $K$ compact there exists $N$ points $p_1, \ldots, p_N$ of $K$ such that the balls $B_\varepsilon(p_i)$ cover the whole set $K$. Let $K_\varepsilon$ be the convex hull of $p_1, \ldots, p_N$ and let $V_\varepsilon$ be the affine $N-1$ dimensional space containing these points so that $K_\varepsilon \subset V_\varepsilon$.

Now consider a projection $\pi_\varepsilon: X \to V_\varepsilon$ such that $\|\pi_\varepsilon(x) - \pi_\varepsilon(y)\| \leq \|x - y\|$ and define $f_\varepsilon: K_\varepsilon \to K_\varepsilon$, $f_\varepsilon(x) = \pi_\varepsilon(f(x))$.

This is a continuous function defined on a convex and compact set $K_\varepsilon$ of a finite dimensional vector space $V_\varepsilon$. Hence by Brouwer fixed point theorem it admits a fixed point $x_\varepsilon$

$$f_\varepsilon(x_\varepsilon) = x_\varepsilon.$$

Since $K$ is sequentially compact we can find a sequence $\varepsilon_k \to 0$ such that $x_k = x_{\varepsilon_k}$ converges to some point $\bar{x} \in K$.

We claim that $f(\bar{x}) = \bar{x}$.

Clearly $f_\varepsilon(x_k) = x_k \to \bar{x}$. To conclude the proof we only need to show that also $f_\varepsilon(x_k) \to f(\bar{x})$ or, which is the same, that $\|f_\varepsilon(x_k) - f(\bar{x})\| \to 0$.

In fact we have

$$\|f_\varepsilon(x_k) - f(\bar{x})\| = \|\pi_\varepsilon(f(x_k)) - f(\bar{x})\|$$

$$\leq \|\pi_\varepsilon(f(x_k)) - f(x_k)\| + \|f(x_k) - f(\bar{x})\|$$

$$\leq \varepsilon_k + \|f(x_k) - f(\bar{x})\| \to 0$$

where we used the fact that $\|\pi_\varepsilon(x) - x\| \leq \varepsilon$ being $x \in K$ contained in some ball $B_\varepsilon$ centered on $K_\varepsilon$. 

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