Schauder's Fixed Point Theorem

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Theorem 0.1. *Let X be a locally convex topological vector space, and let $K \subset X$ be a non-empty, compact, and convex set. Then given any continuous mapping $f\colon K \to K$ there exists $x \in K$ such that $f(x) = x$.*

Notice that the unit disc of a finite dimensional vector space is always convex and compact hence this theorem extends Brouwer Fixed Point Theorem.

Proof. We suppose that X is a Banach space. The idea of the proof is to reduce ourselves to the finite dimensional case. Given $\epsilon > 0$ notice that the family of open sets $\{B_\epsilon(x)\colon x \in K\}$ is an open covering of K. Being K compact there exists a finite subcover, i.e. there exists N points $p_1, \dots, p_N$ of K such that the balls $B_\epsilon(p_i)$ cover the whole set K. Let K_ϵ be the convex hull of $p_1, \dots, p_N$ and let V_ϵ be the affine $N-1$ dimensional space containing these points so that $K_\epsilon \subset V_\epsilon$.

Now consider a projection $\pi_\epsilon\colon X \to V_\epsilon$ such that $\|\pi_\epsilon(x) - \pi_\epsilon(y)\| \le \|x - y\|$ and define

$$f_\epsilon\colon K_\epsilon \to K_\epsilon, \quad f_\epsilon(x) = \pi_\epsilon(f(x)).$$

This is a continuous function defined on a convex and compact set K_ϵ of a finite dimensional vector space V_ϵ. Hence by Brouwer fixed point theorem it admits a fixed point x_ϵ

$$f_\epsilon(x_\epsilon) = x_\epsilon.$$

Since K is sequentially compact we can find a sequence $\epsilon_k \to 0$ such that $x_k = x_{\epsilon_k}$ converges to some point $\bar{x} \in K$.

We claim that $f(\bar{x}) = \bar{x}$.

Clearly $f_{\epsilon_k}(x_k) = x_k \to \bar{x}$. To conclude the proof we only need to show that also $f_{\epsilon_k}(x_k) \to f(\bar{x})$ or, which is the same, that $\|f_{\epsilon_k}(x_k) - f(\bar{x})\| \to 0$.

In fact we have

$$\|f_{\epsilon_k}(x_k) - f(\bar{x})\| = \|\pi_{\epsilon_k}(f(x_k)) - f(\bar{x})\|$$

$$\le \|\pi_{\epsilon_k}(f(x_k)) - f(x_k)\| + \|f(x_k) - f(\bar{x})\|$$

$$\le \epsilon_k + \|f(x_k) - f(\bar{x})\| \to 0$$

where we used the fact that $\|\pi_\epsilon(x) - x\| \le \epsilon$ being $x \in K$ contained in some ball B_ϵ centered on K_ϵ. $\square$