Fourier transform, heat, Poisson and Laplace equations.

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1 The physical problem of heat conduction

Heat conduction is the transfer of heat from warm areas to cooler ones, and effectively occurs by diffusion. The heat flux is therefore

$$\Phi_Q \sim \frac{T_{hot} - T_{cold}}{d} \kappa \rho C_P, \tag{1.1}$$

where ρ is the density of the material, C_P is the mass heat capacity, d is the diffusion distance, and κ is the thermal diffusivity.

Then the thermal conductivity is defined as:

$$k := \kappa \rho C_P. \tag{1.2}$$

Noting that $\frac{T_{hot}-T_{cold}}{d}$ is (minus) the temperature gradient, equation (1.1) becomes Fourier's law:

$$\Phi_Q = -k \nabla T, \tag{1.3}$$

where

$$\nabla T := [\partial_x T, \partial_y T, \partial_z T] \tag{1.4}$$

is the usual gradient vector, and where we assume that the temperature T is a function of both time and position:

$$T = T(x, y, z, ; t).$$

The time-dependent heat conduction equation is given by

$$\partial_t T = \frac{H}{C_P} + \frac{1}{\rho C_P} \nabla \cdot (k \nabla T), \tag{1.5}$$

where H is the heat production per unit mass. In the special case when the termal conductivity k is a constant, the above equation simplifies to

$$\partial_t T = \frac{H}{C_P} + \kappa \Delta T, \tag{1.6}$$

where κ is again the termal diffusivity, and Δ is the Laplace operator acting on the position variable:

$$\Delta T := \partial_x^2 T + \partial_y^2 T + \partial_z^2 T. \tag{1.7}$$

A special case is the one in which H is time independent in the interval $t \geq 0$. This models a welding machine which starts pumping heat into the material, at a constant rate. Thus H is only a function of $[x, y, z]$, and is different from zero only in the contact region between the machine and material.

One can prove that regardless of the initial conditions for $T(x, y, z; t)$, when the time t becomes very large, the temperature distribution $T(x, y, z; t)$ converges toward a dynamic equilibrium state. This means that the temperature gradient in our material reached that particular distribution which dissipates the heat pumped by the machine without varying in time. Mathematically, this means that we reached a time independent stationary solution $T_s(x, y, z)$, which solves a simpler equation:

$$-\Delta T_s(x, y, z) = \frac{H(x, y, z)}{\kappa C_P}. \tag{1.8}$$

We denote vectors from now on with boldface letters; for example, $\mathbf{r} \in \mathbb{R}^3$ denotes the coordinate vector $[x, y, z]$. Without loss of generality, we can assume that H is different from zero only in a ball of radius 1 near the origin of coordinates. This models a finite contact region between the welding machine and material.

In order to solve the above equation, we need only one more thing: the value of T_s at "infinity", that is far away from the welding process. This value T_e is a constant given by the problem. Therefore, if we denote by $\psi(\mathbf{r}) := T_s(\mathbf{r}) - T_e$, we arrive at the equation we are mainly interested in:

$$-\Delta \psi(\mathbf{r}) = \frac{H(\mathbf{r})}{\kappa C_P}, \qquad \lim_{|\mathbf{r}| \to \infty} \psi(\mathbf{r}) = 0. \tag{1.9}$$

This is a second order elliptic partial differential equation called the Poisson equation. If $H = 0$, it reduces to Laplace equation.

2 Solving the Poisson equation

It is worth noting that with the boundary condition we imposed on equation (1.9), it does NOT have a solution in one dimension.

Exercise 2.1. *Consider on the real line the equation $\psi''(x) = g(x)$, where g is a constant g_0 on the interval $[-1, 1]$, and $g = 0$ outside the interval $[-1, 1]$. Assume the boundary condition $\psi(\pm\infty) = 0$. Show that we have a solution if and only if $g_0 = 0$, and then $\psi \equiv 0$.*

Hint. Assume that there is a solution ψ to our equation. Then outside the interval $[-1, 1]$ it must solve the equation $\psi''(x) = 0$. The most general solution to this equation is $\psi(x) = c_1 x + c_2$, where c_1 and c_2 are constants. Because $\psi(\infty) = 0$, we must have $c_1 = 0$ otherwise the limit value would be $\pm\infty$. Hence $\psi(x) = c_2$ outside the interval, hence $c_2 = 0$ due to the boundary value at infinity.

It means that $\psi(-1) = \psi(1) = 0$ and $\psi'(-1) = \psi'(1) = 0$. Then inside the interval $[-1, 1]$ the solution for ψ would be (we integrate twice starting from -1 to the right)

$$\psi(x) = \psi(-1) + (x+1)\psi'(-1) + \frac{(x+1)^2}{2} g_0, \quad x \in [-1, 1].$$

Hence $\psi(x) = \frac{(x+1)^2}{2} g_0$ inside the interval $[-1, 1]$, and in particular $\psi(1) = 2g_0$. Since $\psi(1) = 0$, we get $g_0 = 0$. $\square$

REMARK: the same conclusion holds even if g is not just a constant on $[-1, 1]$, but any smooth enough function. In fact, the same negative conclusion holds in two dimensions, too, but the proof is much more complicated and uses the Green function techniques which will be developed in the next subsection.

2.1 The auxiliary Helmholtz equation

For any $\epsilon \neq 0$ we introduce an auxiliary equation

$$(-\Delta + \epsilon^2)\psi_\epsilon(\mathbf{r}) = \frac{H(\mathbf{r})}{\kappa C_P}, \qquad \lim_{|\mathbf{r}|\to\infty} \psi_\epsilon(\mathbf{r}) = 0, \tag{2.1}$$

and try to solve for ψ_ϵ.

The main tool in solving this equation will be the continuous Fourier transform. In what follows, we enumerate a few fundamental properties of it, including its definition. We let here the dimension to be arbitrary, $n \geq 1$.

Definition 2.2. *Take a smooth function f which is zero outside a large ball in $\mathbb{R}^n$. Then its Fourier transform is defined as the function*

$$(\mathcal{F}[f])(\mathbf{k}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) d\mathbf{r}, \quad \mathbf{k} \in \mathbb{R}^n,$$

where $\mathbf{k}\cdot\mathbf{r} = k_1x_1 + k_2x_2 + ... + k_nx_n$ is the usual dot product in $\mathbb{R}^n$.

The inverse Fourier transform is defined in a similar way:

$$(\mathcal{F}^{-1}[g])(\mathbf{r}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{k}\cdot\mathbf{r}} g(\mathbf{k}) d\mathbf{k}.$$

A first fundamental property is that these mappings are inverses for each other. This means

$$(\mathcal{F}[\mathcal{F}^{-1}[g]])(\mathbf{k}) = g(\mathbf{k}), \quad (\mathcal{F}^{-1}[\mathcal{F}[f]])(\mathbf{r}) = f(\mathbf{r}). \tag{2.2}$$

Another property is that the Fourier transform changes derivatives in one variable with multiplication is the other one. This is shown in the following exercise:

Exercise 2.3. *Show that if $\partial_j f$ denotes the partial derivative of f with respect to the jth variable, we have:*

$$(\mathcal{F}[-i\partial_j f](\mathbf{k}) = k_j(\mathcal{F}[f])(\mathbf{k}), \quad j \in \{1, ..., n\}. \tag{2.3}$$

Moreover, prove that

$$(\mathcal{F}[-\Delta f](\mathbf{k}) = \mathbf{k}^2(\mathcal{F}[f])(\mathbf{k}). \tag{2.4}$$

Hint. Notice the trivial identity

$$k_j e^{-i\mathbf{k}\cdot\mathbf{r}} = i\partial_j e^{-i\mathbf{k}\cdot\mathbf{r}},$$

and use integration by parts. We get

$$k_j(\mathcal{F}[f])(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \{i\partial_j e^{-i\mathbf{k}\cdot\mathbf{r}}\} f(\mathbf{r}) d\mathbf{r} = (\mathcal{F}[-i\partial_j f])(\mathbf{k}),$$

where we also used that f is zero outside a certain region in $\mathbb{R}^n$. If we apply this twice, we get:

$$k_j^2(\mathcal{F}[f])(\mathbf{k}) = (\mathcal{F}[-\partial_j^2 f])(\mathbf{k}),$$

which immediately leads to (2.4). □

The third property is related to the convolution.

Definition 2.4. *The convolution of f and g is defined as*

$$(f * g)(\mathbf{r}) := \int_{\mathbb{R}^n} f(\mathbf{r} - \mathbf{r}')g(\mathbf{r}')d\mathbf{r}'.$$

By a change of variable, we see that the convolution is commutative, i.e. $f * g = g * f$.

Exercise 2.5. *Show that the Fourier transform sends a convolution into multiplication:*

$$(\mathcal{F}[f * g](\mathbf{k}) = (2\pi)^{n/2}(\mathcal{F}[f])(\mathbf{k})(\mathcal{F}[g])(\mathbf{k}). \tag{2.5}$$

Moreover, prove that

$$\mathcal{F}^{-1}[\mathcal{F}[f]\mathcal{F}[g]] = (2\pi)^{-n/2} f * g. \tag{2.6}$$

Hint. According to the definition, we have

$$(\mathcal{F}[f * g](\mathbf{k}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot\mathbf{r}} \left(\int_{\mathbb{R}^n} f(\mathbf{r} - \mathbf{r}')g(\mathbf{r}')d\mathbf{r}' \right) d\mathbf{r} \tag{2.7}$$

$$= (2\pi)^{n/2} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} d\mathbf{r}' \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} f(\mathbf{r} - \mathbf{r}')d\mathbf{r} \right) e^{-i\mathbf{k}\cdot\mathbf{r}'} g(\mathbf{r}')d\mathbf{r}'$$

where we interchanged the order of integrals, and wrote $\mathbf{k}\cdot\mathbf{r} = \mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') + \mathbf{k}\cdot\mathbf{r}'$. Now the integral with respect to $\mathbf{r}$ will give (after a change of variable) $(\mathcal{F}[f])(\mathbf{k})$, and finally we perform the integral over $\mathbf{r}'$ and get (2.5). The formula (2.6) is obtained from (2.5) by applying (2.2). □

We are now in position of solving the modified Poisson equation (2.1). As long as $\epsilon \neq 0$, we can work in any dimensions $n \geq 1$, not just in three. Take the Fourier transform in both sides and use (2.4). We get

$$(\mathcal{F}[-\Delta\psi_\epsilon + \epsilon^2\psi_\epsilon])(\mathbf{k}) = (\mathbf{k}^2 + \epsilon^2)(\mathcal{F}[\psi_\epsilon])(\mathbf{k}) = \frac{1}{\kappa C_P}(\mathcal{F}[H])(\mathbf{k}),$$

therefore

$$(\mathcal{F}[\psi_\epsilon])(\mathbf{k}) = \frac{1}{\kappa C_P}\frac{1}{\mathbf{k}^2 + \epsilon^2}\mathcal{F}[H])(\mathbf{k}). \tag{2.8}$$

Now denote by

$$G_\epsilon(\mathbf{r}) := \frac{1}{(2\pi)^{n/2}}(\mathcal{F}^{-1}[1/(\mathbf{k}^2 + \epsilon^2)])(\mathbf{r}). \tag{2.9}$$

We see that (2.8) can be rewritten as

$$(\mathcal{F}[\psi_\epsilon])(\mathbf{k}) = \frac{1}{\kappa C_P}(2\pi)^{n/2}(\mathcal{F}[G_\epsilon])(\mathbf{k})\mathcal{F}[H])(\mathbf{k}).$$

If we apply the inverse Fourier transform in both sides, and use (2.6) in the right hand side, we get:

$$\psi_\epsilon(\mathbf{r}) = \frac{1}{\kappa C_P}(G_\epsilon * H)(\mathbf{r}) = \int_{\mathbb{R}^n} G_\epsilon(\mathbf{r} - \mathbf{r}')\frac{H(\mathbf{r}')}{\kappa C_P}d\mathbf{r}'. \tag{2.10}$$

This is a fundamental formula which gives the inverse of the Helmholtz operator $-\Delta + \epsilon^2$, via its "Green function" G_ϵ. It holds for all dimensions; troubles arise only when one takes ϵ to zero.

One can compute G_ϵ from formula (2.9), and obtain the general formula:

$$G_\epsilon(\mathbf{r}) = \frac{1}{2\pi}\left(\frac{\epsilon}{2\pi|\mathbf{r}|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(\epsilon|\mathbf{r}|),$$

where $K_\nu(z)$ are the Macdonald functions; for more details, see for example the webpage

http : //mathworld.wolfram.com/ModifiedBesselFunctionoftheSecondKind.html

It is worth noticing that when $n = 1$ we have

$$G_\epsilon(\mathbf{r}) = \frac{1}{2\epsilon}e^{-\epsilon|\mathbf{r}|},$$

while for $n = 3$ we have

$$G_\epsilon(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}|}e^{-\epsilon|\mathbf{r}|}.$$

2.2 The solution to the Poisson equation

Assume that $n = 3$; then according to the previous subsection we have

$$\psi_\epsilon(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{e^{-\epsilon|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \frac{H(\mathbf{r}')}{\kappa C_P} d\mathbf{r}'.$$

Now define

$$\psi(\mathbf{r}) := \lim_{\epsilon \to 0} \psi_\epsilon(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \frac{H(\mathbf{r}')}{\kappa C_P} d\mathbf{r}'. \tag{2.11}$$

After integration by parts, one easily obtains that

$$\Delta\psi_\epsilon(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{e^{-\epsilon|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \frac{\Delta H(\mathbf{r}')}{\kappa C_P} d\mathbf{r}'$$

and

$$\Delta\psi(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \frac{\Delta H(\mathbf{r}')}{\kappa C_P} d\mathbf{r}'. \tag{2.12}$$

Hence we also have:

$$\lim_{\epsilon \to 0} \Delta\psi_\epsilon(\mathbf{r}) = \Delta\psi(\mathbf{r}). \tag{2.13}$$

Therefore, by taking the limit $\epsilon \to 0$ in (2.1), and using (2.11) and (2.13) we obtain

$$-\Delta\psi(\mathbf{r}) = \frac{H(\mathbf{r})}{\kappa C_P}.$$

Moreover, since for $|\mathbf{r}|$ large, $\psi(\mathbf{r})$ behaves like $1/|\mathbf{r}|$, we observe that the boundary condition at infinity is also fulfilled, and the Poisson equation (1.9) solved.

REMARK: we see that for $n = 1$ we cannot repeat this argument, because G_ϵ diverges when $\epsilon \to 0$.

2.3 When H is a delta-Dirac distribution

Now assume that the contact area between the welding machine and the material becomes smaller and smaller, while the heat rate pumping stays the same. Mathematically this means that while the integral

$$h_0 := \int_{\mathbb{R}^3} H(\mathbf{r}')d\mathbf{r}'$$

remains constant, the region where H is different from zero shrinks to a small ball around the origin of coordinates. At the limit, for every smooth function f we have:

$$\int_{\mathbb{R}^3} f(\mathbf{r}')H(\mathbf{r}')d\mathbf{r}' \approx h_0 f(0,0,0) \;'' ='' \int_{\mathbb{R}^3} f(\mathbf{r}')h_0\delta(\mathbf{r}')d\mathbf{r}'.$$

Therefore, the solution to (1.9) gets close to

$$\psi(\mathbf{r}) \approx \frac{1}{4\pi|\mathbf{r}|} \frac{h_0}{\kappa C_P},$$

while the stationary temperature distribution in our welding model becomes

$$T_s(\mathbf{r}) \approx T_e + \frac{1}{4\pi|\mathbf{r}|}\frac{h_0}{\kappa C_P}.$$

3 When the welding machine is moving

We now look at the case when H is time dependent. More precisely, we assume that

$$H(x, y, z; t) = H_v(x - vt, y, z), \tag{3.1}$$

which models a translation on the x axis with a constant positive speed $v > 0$. Then we are interested in a particular solution to equation (1.6), where T looks like

$$T(x, y, z; t) = T_s(x - vt, y, z). \tag{3.2}$$

Denote by $\xi = x - vt$.

Exercise 3.1. *Show that* $\partial_t T(\mathbf{r}; t) = -v(\partial_\xi T_s)(\xi, y, z)$, *and* $\Delta T(\mathbf{r}; t) = \Delta T_s(\xi, y, z)$.

Hint. Use the chain rule. □

Therefore, the heat equation (1.6) becomes

$$-\Delta T_s(\xi, y, z) - \frac{v}{\kappa}\partial_\xi T_s(\xi, y, z) = \frac{H_v(\xi, y, z)}{\kappa C_p}. \tag{3.3}$$

Introduce a new unknown function

$$u_s(\xi, y, z) := T_s(\xi, y, z) - T_e, \tag{3.4}$$

where T_e is the equilibrium temperature, far from the welding region. The equation for u_s that we have to solve becomes

$$-\Delta u_s - i\frac{v}{\kappa}(-i\partial_\xi u_s) = \frac{H_v(\xi, y, z)}{\kappa C_p}, \qquad \lim_{\sqrt{\xi^2+y^2+z^2}\to\infty} u_s = 0. \tag{3.5}$$

As in the previous section, we first look at a related equation

$$(-\Delta + \epsilon^2)\psi_\epsilon - i\frac{v}{\kappa}(-i\partial_\xi \psi_\epsilon) = \frac{H_v(\xi, y, z)}{\kappa C_p}, \qquad \lim_{\sqrt{\xi^2+y^2+z^2}\to\infty} \psi_\epsilon = 0. \tag{3.6}$$

Now take the Fourier transform in both sides, and use (2.3) and (2.4). It gives

$$\left\{\mathbf{k}^2 + \epsilon^2 - i\frac{vk_1}{\kappa}\right\}(\mathcal{F}[\psi_\epsilon])(\mathbf{k}) = \frac{(\mathcal{F}[H_v])(\mathbf{k})}{\kappa C_p}.$$

We have

$$\mathbf{k}^2 - i\frac{vk_1}{\kappa} + \epsilon^2 = \left(k_1 - \frac{iv}{2\kappa}\right)^2 + k_2^2 + k_3^2 + \frac{v^2}{4\kappa^2} + \epsilon^2.$$

Reasoning as we did for (2.10), we can write down the solution as a convolution with a Green function equal to

$$G_\epsilon(\xi, y, z) = \frac{1}{(2\pi)^{3/2}} \mathcal{F}^{-1} \left[\frac{1}{\left(k_1 - \frac{iv}{2\kappa}\right)^2 + k_2^2 + k_3^2 + \frac{v^2}{4\kappa^2} + \epsilon^2} \right] (\xi, y, z). \quad (3.7)$$

One can show that this inverse Fourier transform equals:

$$G_\epsilon(\xi, y, z) = \frac{1}{4\pi\sqrt{\xi^2 + y^2 + z^2}} e^{-\frac{\xi v}{2\kappa}} e^{-\alpha\sqrt{\xi^2+y^2+z^2}}, \quad \alpha := \sqrt{\epsilon^2 + v^2/(4\kappa^2)}. \quad (3.8)$$

Define

$$G_0(\xi, y, z) := \frac{1}{4\pi\sqrt{\xi^2 + y^2 + z^2}} e^{-\frac{\xi v}{2\kappa}} e^{-\frac{v}{2\kappa}\sqrt{\xi^2+y^2+z^2}}. \quad (3.9)$$

At the final end, by taking ϵ to 0, one can prove that ψ_ϵ converges to u_s and we eventually get:

$$u_s(\xi, y, z) = \int_{\mathbb{R}^3} G_0(\xi - x', y - y', z - z') \frac{H_v(x', y', z')}{\kappa C_p} dx'dy'dz'. \quad (3.10)$$

Notice again that u_s behaves like $1/\sqrt{\xi^2 + y^2 + z^2}$ at large distances.

In case when H_v is again a delta-Dirac distribution centred at the origin, we have at last for the stationary solution:

$$T(x, y, z; t) = T_e + \frac{1}{4\pi\sqrt{(x - vt)^2 + y^2 + z^2}} e^{-\frac{(x-vt)v}{2\kappa}} e^{-\frac{v}{2\kappa}\sqrt{(x-vt)^2+y^2+z^2}} \frac{h_0}{\kappa C_p}.$$

If $v = 0$, we recover the result from the previous section.