Integration and Fourier Theory

Lecture 12

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1 The L^p-spaces

Last time, the first part of the following definition was sneaked in through the back door (or rather, through the formulation of the theorem containing Hölder's and Minkowski's inequalites):

Definition 1.1. If $0 < p < \infty$ and f is any complex, measurable function on X, then we write

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

and let $L^p(\mu)$ denote the set of all f for which $\|f\|_p < \infty$. We call $\|\cdot\|_p$ the L^p-norm.

Remark 1.2. As all of you have already noted, if $f = 0$ a.e. (but differs from 0 on a set of measure zero), then $\|f\|_p = \|0\|_p = 0$ although $f \neq 0$. This shows that $\|\cdot\|_p$ is not a proper norm. We will repair this flaw in due course.

In a few important special cases, the notation $L^p(\mu)$ is slightly altered. This includes the case where $X = \mathbb{R}^k$ and μ is the Lebesgue measure on $\mathbb{R}^k$. In this case, we write $L^p(\mathbb{R}^k)$ instead of $L^p(\mu)$. Another example is when μ is the counting measure on a set A, in which case one writes $\ell^p(A)$, or, when A is countable, sometimes just ℓ^p. An element of ℓ^p may be regarded as a complex sequence $x = \{\xi_n\}_{n=1}^{\infty}$ with

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{\frac{1}{p}} < \infty\,.$$

Definition 1.3. For any measurable function $g\colon X \to [0, \infty]$ let

$$\operatorname{ess\,sup} g = \operatorname*{ess\,sup}_{x \in X} \{g(x)\} = \inf\{\alpha \in \mathbb{R} \parallel \mu(g^{-1}((\alpha, \infty])) = 0\}$$

denote the *essential supremum* of g, i.e. the essential supremum of g, $\operatorname{ess\,sup} g$, is the infimum of the set S of real α such that $\mu(g^{-1}((\alpha, \infty])) = 0$. Here, the convention $\inf \varnothing = \infty$ is employed.

If f is a complex, measurable function on X, then let $\|f\|_\infty = \operatorname{ess\,sup}|f|$ denote the *essential bound* or *supremum norm* of f. Let $L^\infty(\mu)$ (or $L^\infty(\mathbb{R}^k)$, $\ell^\infty(A)$ or ℓ^∞, respectively) denote the set of all *essentially bounded*, measurable functions, i.e. all complex, measurable functions f on X for which $\|f\|_\infty < \infty$.

Remark 1.4. Since

$$g^{-1}((\operatorname{ess\,sup} g, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\operatorname{ess\,sup} g + \frac{1}{n}, \infty\right]\right)$$

and each of the countably many sets on the right-hand side has measure zero, we conclude that $\mu(g^{-1}(\operatorname{ess\,sup} g, \infty]) = 0$.

In the case where μ is the counting measure on $X = A$, essentially boundedness and boundedness is the same, since all nonempty sets have positive measure.

The notation $L^\infty(\mu)$ strongly suggests a connection with the L^p-spaces for finite p. This is of course no coincidence and $p = \infty$ may in fact be seen as a limiting case as the following two theorems indicate.

Theorem 1.5. *Let p and q be Hölder conjugates $1 \leqslant p, q \leqslant \infty$ and $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \leqslant \|f\|_p \|g\|_q$$

Proof. For $1 < p, q < \infty$, the statement is simply Hölder's inequality applied to $|f|$ and $|g|$. Assume $p = \infty$. Then by Remark 1.4, we know that $\mu(\{x \in X \parallel f(x) > \|f\|_\infty\}) = 0$, so

$$|f(x)g(x)| \leqslant \|f\|_\infty |g(x)| \qquad \text{a.e. } x \text{ in } X\,. \tag{1}$$

Integrating both sides of (1) yields the result for $p = \infty$ and $q = 1$, and interchanging the rôles of f and g takes care of $p = 1$ and $q = \infty$. □

Theorem 1.6. *Let $1 \leqslant p \leqslant \infty$ and $f, g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$ and*

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p \tag{2}$$

Proof. Note that for $1 \leqslant p < \infty$,

$$|f + g|^p \leqslant (|f| + |g|)^p \tag{3}$$

($x \mapsto x^p$ is an increasing function), so for $1 < p < \infty$, (2) follows from Minkowski's inequality. If $p = \infty$ or $p = 1$, (2) trivially follows from (3) with $p = 1$. □

Remark 1.7. If $1 \leqslant p \leqslant \infty$, $f \in L^p(\mu)$ and $\alpha \in \mathbb{C}$, then clearly $\alpha f \in L^p(\mu)$ with $\|\alpha f\|_p = |\alpha| \|f\|_p$. In addition, by Theorem 1.6, we have that if also $g \in L^p(\mu)$, then $f + g \in L^p(\mu)$. Hence $L^p(\mu)$ is a *complex vector space*.

In fact, in conjunction with $\|\cdot\|_p$, it is *almost* a *normed* complex vector space, but, as already noted in Remark 1.2, $\|f\|_p = 0$ does not imply $f = 0$ (it is a so-called *seminormed* vector space). In the following, we will fix this by slightly redefining the L^p-spaces, use the norm to define a metric, and show that the (redefined) L^p-spaces are complete in this metric.

Independently of whether $1 \leqslant p < \infty$ or $p = \infty$, it follows from the definition of $\|\cdot\|_p$ that if $\|f\|_p = 0$ then $f = 0$ a.e. If we let $\mathcal{Z}$ denote the space of all functions f which are 0 a.e., then the quotient space $L^p(\mu)/\mathcal{Z} = \{f + \mathcal{Z} \mid f \in L^p(\mu)\}$ equipped with the induced norm $|||f + \mathcal{Z}|||_p = \|f\|_p$ satisfies that $|||f + \mathcal{Z}|||_p = 0$ if and only if $f + \mathcal{Z} = 0 + \mathcal{Z}$. We begin by showing that $|||\cdot|||_p$ is well-defined:

Let $f, g \in L^p(\mu)$ be such that $f + \mathcal{Z} = g + \mathcal{Z}$. We need to show that $|||f + \mathcal{Z}|||_p = |||g + \mathcal{Z}|||_p$, i.e. that $\|f\|_p = \|g\|_p$. But this follows from the fact that, since $f + \mathcal{Z} = g + \mathcal{Z}$, we have $f - g \in \mathcal{Z}$, and hence $f = g$ a.e.

To see that, indeed, $|||f + \mathcal{Z}|||_p = 0$ if and only if $f + \mathcal{Z} = 0 + \mathcal{Z}$, we note that the "if" part follows from $|||\cdot|||_p$ being well-defined, and the "only if" from the fact that $\|f\|_p = 0$ only if $f \in \mathcal{Z}$.

Having now established a norm on $L^p(\mu)/\mathcal{Z}$, we define the induced metric d in the obvious way:

$$d(f + \mathcal{Z}, g + \mathcal{Z}) = |||(f + \mathcal{Z}) - (g + \mathcal{Z})|||_p = \|f - g\|_p \,.$$

We want to show that $L^p(\mu)/\mathcal{Z}$ is a *complete* metric space, i.e. that every Cauchy sequence in $L^p(\mu)/\mathcal{Z}$ converges to an element of $L^p(\mu)/\mathcal{Z}$. Before taking on that challenge, however, we need to agree on something: quotient space notation is really cumbersome, and since we in integration theory actually don't really care about what happens on a set of measure 0, we will drop this notation and just redefine $L^p(\mu)$ to be the quotient space, tacitly work with *representatives* of elements in this space, and write "a.e." in cases where the circumstances require it. In the same vein, we will refrain from using $|||\cdot|||_p$ and stick to the familiar $\|\cdot\|_p$.

Before proceeding to the completeness theorem, we recall (and formulate in terms of L^p-spaces) the definition of convergence, Cauchy sequences and completeness:

Definition 1.8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions[1] in $L^p(\mu)$. If there exists a function $f \in L^p(\mu)$ such that $\|f - f_n\|_p \to 0$ as $n \to \infty$, we say that $\{f_n\}$ *converges* to f in $L^p(\mu)$ (or in p-mean or that it is L^p-convergent). If, for all $\varepsilon > 0$, there exists an N such that for all $n, m > N$, we have $\|f_n - f_m\|_p < \varepsilon$, then $\{f_n\}$ is called a *Cauchy sequence* in $L^p(\mu)$.

Obviously, any convergent sequence is a Cauchy sequence. When the reverse statement is true, the underlying space is called *complete*.

Definition 1.9. If any Cauchy sequence on a metric space is convergent, the underlying space is called complete.

Theorem 1.10. *Let $1 \leqslant p \leqslant \infty$ and μ be a positive measure. Then $L^p(\mu)$ is a complete metric space.*

Proof. First assume that $1 \leqslant p < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence. By assumption, there exists a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ with $n_1 < n_2 < n_3 < \cdots$ such that

$$\|f_{n_{i+1}} - f_{n_i}\|_p < \frac{1}{2^i} \,. \tag{4}$$

Put

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, \qquad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|$$

[1] In line with the previous paragraph, we will continue to refer to elements of $L^p(\mu)$ as "functions"

From (4) and the Minkowski inequality, it follows that $\|g_k\|_p < 1$ for all k. Hence we can use Fatou's lemma to get

$$\|g\|_p^p = \int g^p \,\mathrm{d}\mu = \int \liminf_{k\to\infty} g_k^p \,\mathrm{d}\mu \leqslant \liminf_{k\to\infty} \int g_k^p \,\mathrm{d}\mu \leqslant 1 \,.$$

This shows, in particular, that $g(x) < \infty$ a.e., so

$$f_{n_1} + \sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i}) \tag{5}$$

converges absolutely a.e. Define f to be equal to the sum in (5) where it converges absolutely, and 0 elsewhere. Since

$$f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i}) = f_{n_k}$$

we see that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x) \qquad \text{a.e.}$$

We have now found a function f which is the pointwise limit a.e. of a subsequence of $\{f_n\}$. We now have to show that this is the L^p-limit of $\{f_n\}$. Let $\varepsilon > 0$ and let N be such that for $n, m > N$, $\|f_n - f_m\|_p < \varepsilon$. Again we employ Fatou's lemma:

$$\int_X |f - f_m|^p \,\mathrm{d}\mu = \int_X \liminf_{i\to\infty} (|f_{n_i} - f_m|^p) \,\mathrm{d}\mu \leqslant \liminf_{i\to\infty} \int_X |f_{n_i} - f_m|^p \,\mathrm{d}\mu \leqslant \varepsilon^p \,.$$

This shows first that $(f - f_m) \in L^p(\mu)$, secondly, since $f_m \in L^p(\mu)$ and $f = (f - f_m) + f_m$ that $f \in L^p(\mu)$ and, since ε was arbitrary, that $\|f - f_m\|_p \to 0$ as $m \to \infty$.

Now assume that $p = \infty$. Let A_k be the set where $|f_k| > \|f_k\|_\infty$, and $B_{m,n}$ the set where $|f_n - f_m| > \|f_n - f_m\|_\infty$. There are countably many sets, and each of them have measure 0. Hence their union E has measure 0. On the complement E^c of E we have

$$|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_\infty \,, \qquad \forall x \in E^c \,, \tag{6}$$

so $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$ for each $x \in E^c$ and hence convergent to some number $f(x)$. Define a function $f\colon X \to \mathbb{C}$ by $x \mapsto f(x)$ for $x \in E^c$ and $f(x) = 0$ otherwise. The convergence $f_n(x) \mapsto f(x)$ is uniform (see (6)), so $\{f_n\}_{n=1}^\infty$ converges uniformly to f a.e. i.e. $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. $\square$

The following theorem is clear from the previous proof:

Theorem 1.11. *Let $1 \leqslant p \leqslant \infty$ and $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$ with limit f. Then there exists a subsequence $\{f_{n_i}\}$ which converges pointwise a.e. to f.*