Integration and Fourier Theory

Lecture 14

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1 Trigonometric Series

Denote the unit circle in the complex plane by $T = \{x \in \mathbb{C} \mid |x| = 1\}$ ('T' for 'torus'). Note that $\mathbb{R} \ni t \mapsto e^{it} \in T$ is onto. This means that if $F\colon T \to \mathbb{C}$ is a function on T, then the function $f\colon \mathbb{R} \to \mathbb{C}$ defined by

$$f(t) = F(e^{it})$$

is a 2π-periodic function containing all information on F. Likewise, for any 2π-periodic function f, i.e. a function f satisfying $f(t) = f(t + 2\pi)$ for all t, there exists a function $F\colon T \to \mathbb{C}$ such that $f(t) = F(e^{it})$. This means that we can identify 2π-periodic functions f on $\mathbb{R}$ with functions on T, and, for simplicity of notation, we shall sometimes write $f(t)$ instead of $f(e^{it})$, even if we think of f as being defined on T.

With these conventions in mind, we define $L^p(T)$ for $1 \leqslant p \leqslant \infty$:

Definition 1.1. Let $1 \leqslant p < \infty$. Then we define $L^p(T)$ to be the set of all Lebesgue measurable, 2π-periodic functions $f\colon \mathbb{R} \to \mathbb{C}$ with norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty\,. \tag{1}$$

The factor $\frac{1}{2\pi}$ is just to normalize the measure of $[-\pi, \pi]$ to 1, i.e. ensure that the constant function $f(t) = 1$ integrates to 1, a fact that will simplify certain things in the following.

Definition 1.2. For $p = \infty$, we let $L^p(T)$ denote the set of 2π-periodic members of $L^\infty(\mathbb{R})$ equipped with the usual essential supremum norm. We let $C(T)$ denote the set of all complex, continuous 2π-periodic functions, and also equip this space with the supremum norm (the word "essential" can be skipped here because of the continuity).

Note that also these two spaces contain the constant function $f(t) = 1$ and that its norm is again 1.

Definition 1.3. A *trigonometric polynomial* is a function $f\colon \mathbb{R} \to \mathbb{C}$ of the following form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(nt) + b_n \sin(nt)) , \tag{2}$$

where $a_0, a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ are complex numbers.

By using Euler's identities, one sees that (2) can also be written in the form

$$f(t) = \sum_{n=-N}^{N} c_n e^{int} ,$$

which is usually a more convenient form. Note that all trigonometric polynomials are in $L^p(T)$ for any $1 \leqslant p \leqslant \infty$ and they are also in $C(T)$.

$L^2(T)$ with the norm (1) is a Hilbert space, and the inner product is

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, \mathrm{d}t .$$

If we now, for every $n \in \mathbb{Z}$, define:

$$u_n(t) = e^{int} ,$$

then

$$(u_n, u_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t}\, \mathrm{d}t = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} ,$$

as a quick calculation shows. In other words, the set $\{u_n \in L^2(T) \mid n \in \mathbb{Z}\}$ is an orthonormal system in $L^2(T)$. In fact, it turns out to be complete.

2 Completeness of the trigonometric system I

We saw last time how completeness of an orthonormal system was equivalent to density of its span. We clearly have the following inclusions: $\operatorname{span}\{u_n\}_{n\in\mathbb{Z}} \subset C(T) \subset L^2(T)$, and by [Rudin, Theorem 3.14], the latter inclusion is dense (in L^2-sense). We thus only need to show that the former inclusion also is dense in L^2-sense, i.e. that to every $f \in C(T)$ and every $\varepsilon > 0$, there exists a trigonometric polynomial $P \in \operatorname{span}\{u_n\}_{n\in\mathbb{Z}}$ such that $\|f - P\|_2 < \varepsilon$. But since the natural norm on $C(T)$, $\|\cdot\|_\infty$, satisfies $\|g\|_2 \leqslant \|g\|_\infty$ for every $g \in C(T)$, we are done if we can show that $\|f - P\|_\infty < \varepsilon$, i.e. that $\operatorname{span}\{u_n\}_{n\in\mathbb{Z}}$ is dense in $C(T)$ in $\|\cdot\|_\infty$-sense. We will show this using something called an *approximation to the identity*, a concept of a more general nature. However, we will only introduce a specific version which is suitable in the present setup.

Remark 2.1. The alert student has noticed that $C(T)$ is a space of functions, while $L^2(T)$ is a space of classes of functions. It should be obvious, however, how to interpret the statement that $C(T) \subset L^2(T)$. The only thing one should be careful about noticing is that whenever one works with a continuous function (or a class of functions in L^2 of which a continuous representative can be chosen), we will refrain from the "a.e." and not hesitate to consider the value of the (continuous representative of the) function (class) in specific points.

3 Approximation to the identity

Last time, in the self-study part of Lecture 13, you learned that every continuous, linear functional on a Hilbert space H is of the form $H \ni x \mapsto (x, y)$ for some $y \in H$ ([Rudin, Theorem 4.12]). Now consider $C(T) \subset L^2(T)$ and the obviously linear functional $C(T) \ni f \mapsto f(0) \in \mathbb{C}$.

It is easy to see that it is continuous (on $C(T)$ equipped with the supremum norm), but also clear that one cannot hope to extend it to a continuous, linear functional on $L^2(T)$, if not for other reasons, then because $f(0)$ is not in general well-defined for a function $f \in L^2(T)$. This means that we cannot hope to find an element $g \in L^2(T)$ such that $(f, g) = f(0)$, not even if f is continuous.

An *approximation to the identity* on $L^2(T)$ is something which in a sense comes as close as possible to doing what we wanted of g above. To touch upon why it is called an approximation to the *identity*, we note that one can define a so-called *convolution* (usually denoted by $*$) between two functions f, g (at least if they are both in $L^1(T)$) by

$$f * g(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) g(s) \, \mathrm{d}s \,. \tag{3}$$

Note that the assumed 2π-periodicity of f ensures that even though $t - s$ might not lie in $[-\pi, \pi]$, it is still enough to specify f on this interval (and changes of variables in the integral are easy).

Now *if* we had a $g \in L^2(T) \cap L^1(T)$ which satisfied that $(f, g) = f(0)$, then we would have that

$$f * \overline{g}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) \overline{g(s)} \, \mathrm{d}s = (f(t - \cdot), g) = f(t-0) = f(t) \,.$$

This means that in the triple $(L^1(T), +, *)$, which has the algebraic property of being a *ring*, $\overline{g}$ would be an *identity*. Having already established the nonexistence of such an element, the above discussion may seem to be overly hypothetical. Getting back to the real world, we will now show that if a sequence of functions $\{k_n\}_{k=1}^{\infty}$ has the property of being an approximation to the identity (defined below), then for $f \in C(T)$, we have that $(f, k_n) \to f(0)$ when $n \to \infty$.

Definition 3.1 (Approximation of the identity)**.** A sequence $\{k_n\}_{n=1}^{\infty}$ is called an *approximation of the identity* if the following holds:

(a) For every n, we have $k_n \geqslant 0$.

(b) We have $k_n \in L^1(T) \cap L^\infty(T)$ with $\frac{1}{2\pi} \int_{-\pi}^{\pi} k_n(t) \, \mathrm{d}s = 1$ for all n.

(c) If for every $\delta > 0$, $D_\delta = \{t \mid \pi \geqslant |t| \geqslant \delta\}$, then $\|\chi_{D_\delta} k_n\|_\infty \to 0$ as $n \to \infty$.

Remark 3.2. The above assumptions means that the weights of the functions k_n concentrate on an increasingly smaller neighborhood of 0. The following things are worth noting:

1. We have that $k_n = \overline{k_n}$ by (a).
2. It can be proved that (b) implies that $k_n \in L^2(T)$.
3. In other words, (c) can be formulated as: k_n converges uniformly to 0 on any set bounded away from 0. As $[-\pi, \pi]$ is compact, it implies that $\|\chi_{D_\delta} k_n\|_1 \to 0$ when $n \to \infty$. Had we been working on $(-\infty, \infty)$, we would have needed to assume this as a fourth condition.

Theorem 3.3. *Let $f \in C(T)$ and $\{k_n\}_{n=1}^{\infty}$ be an approximation to the identity. Then* $\lim_{n\to\infty} \|f - f * k_n\|_\infty = 0$.

Remark 3.4. So far, we have only stated (and not proved) that $f * g$ is well-defined if $f, g \in L^1(T)$. However, from the assumptions on f and k_n, it is clear that (3) defines an element of $L^\infty(T)$, so $\|f - f * k_n\|_\infty$ is well-defined and finite for every n.

Corollary 3.5. *Let $f \in C(T)$ and $\{k_n\}_{n=1}^\infty$ be an approximation to the identity. Then* $\lim_{n\to\infty}(f, k_n) = f(0)$.

Proof of Theorem 3.3. Let $\varepsilon > 0$ and $f \in C(T)$ be given. Since f is uniformly continuous on T, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Since $\|k_n\|_1 = 1$, we have

$$f(t) - f * k_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(t) - f(t-s)\big)k_n(s)\,\mathrm{d}s\,.$$

Since $k_n \geqslant 0$, we can write

$$\begin{aligned}|f(t) - f * k_n(t)| &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t) - f(t-s)|k_n(s)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(t) - f(t-s)|k_n(s)\,\mathrm{d}s + \frac{1}{2\pi}\int_{\pi\geqslant|t|\geqslant\delta}|f(t) - f(t-s)|k_n(s)\,\mathrm{d}s\,. \qquad (4)\end{aligned}$$

Call the two terms in (4) for A_1 and A_2. Then by the choice of δ, we have

$$A_1 = \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(t) - f(t-s)|k_n(s)\,\mathrm{d}s < \varepsilon\int_{-\delta}^{\delta}k_n(s)\,\mathrm{d}s \leqslant \varepsilon\|k_n\|_1 = \varepsilon\,.$$

In A_2, we have that $\|\chi_{D_\delta}k_n\|_\infty \to 0$. Hence, by the triangle inequality, we have

$$A_2 = \frac{1}{2\pi}\int_{\pi\geqslant|t|\geqslant\delta}|f(t) - f(t-s)|k_n(s)\,\mathrm{d}s \leqslant 2\|f\|_\infty\|\chi_{D_\delta}k_n\|_\infty < \varepsilon$$

for n sufficiently large. This means that

$$|f(t) - f * k_n(t)| < 2\varepsilon$$

for the same sufficiently large n, independently of t, and we are done. □

4 Completeness of the trigonometric system II

What is left to show is that: (1) we can find an approximation to the identity $\{k_n\}_{n=1}^\infty$ which consists of trigonometric polynomials, and (2) when k_n is a trigonometric polynomial, so is $f * k_n$.

Ad (1): put

$$k_n(t) = c_n\left(\frac{1 + \cos(t)}{2}\right)^n,$$

where c_n is chosen so that $\|k_n\|_1 = 1$. Then k_n is obviously a trigonometric polynomial and $k_n \geqslant 0$. It remains to show that $\|\chi_{D_\delta}k_n\|_\infty \to 0$. Since k_n is even, we have

$$1 = \|k_n\|_1 = \frac{c_n}{\pi}\int_0^\pi\left(\frac{1 + \cos(t)}{2}\right)^n\mathrm{d}t > \frac{c_n}{\pi}\int_0^\pi\left(\frac{1 + \cos(t)}{2}\right)^n\sin(t)\,\mathrm{d}t = \frac{2c_n}{\pi(n+1)}\,,$$

where we used that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1+\cos(t)}{2}\right)^{n+1} = -\frac{n+1}{2}\left(\frac{1+\cos(t)}{2}\right)^{n}\sin(t)\,.$$

But then $c_n < \frac{\pi(n+1)}{2}$. Since k_n is decreasing on $[0,\pi]$, we have that

$$k_n(t) \leqslant k_n(\delta) \leqslant \frac{\pi(n+1)}{2}\left(\frac{1+\cos(\delta)}{2}\right)^{n} \qquad \text{for } 0 < \delta \leqslant |t| \leqslant \pi,$$

which shows that $\|\chi_{D_\delta} k_n\|_\infty \to 0$ as $n \to \infty$.

Ad (2): let p be a trigonometric polynomial. Then p can be written as

$$p(t) = \sum_{n=-N}^{N} a_n e^{int}.$$

But, for $f \in C(T)$, we have

$$\begin{aligned} f * p(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)p(s)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+s)p(-s)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)p(t-s)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\sum_{n=-N}^{N} a_n e^{in(t-s)}\,\mathrm{d}s \\ &= \sum_{n=-N}^{N} a_n e^{int}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{-ins}\,\mathrm{d}s, \end{aligned}$$

which is clearly a trigonometric polynomial.

We have thus proved the following important result:

Theorem 4.1. *If $f \in C(T)$ and $\varepsilon > 0$, then there is a trigonometric polynomial P such that*

$$|f(t) - P(t)| < \varepsilon$$

for every real t.

By the discussion in Section 2, this means that

Corollary 4.2. *The trigonometric polynomials are dense in $L^2(T)$ and $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(T)$.*