Notes related to the vector analysis part of the course

Horia D. Cornean

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Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, 9220 Aalborg, Denmark.

1 Path integrals

1.1 General things about paths

A three dimensional path, or curve, is parameterized by just one real variable. Mathematically, a path is the range of a function

$$[a,b] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3.$$

Some important examples:

- A straight line starting at point $\vec{r}_A := x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k}$ and ending at point $\vec{r}_B := x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}$:

$$[0,1] \ni u \mapsto \vec{r}(u) = [(1-u)x_A + ux_B]\mathbf{i} + [y_A(1-u) + y_B u]\mathbf{j} + [z_A(1-u) + z_B u]\mathbf{k}. \tag{1.1}$$

  We denote it by $L_{A\to B}$.

- A straight line starting at point A and going to B, then continued from B to C:

$$[0,2] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3, \tag{1.2}$$

$$\vec{r}(u) = \begin{cases} [x_A(1-u) + x_B u]\mathbf{i} + [y_A(1-u) + y_B u]\mathbf{j} + [z_A(1-u) + z_B u]\mathbf{k}, & u \in [0,1) \\ [x_B(2-u) + x_C(u-1)]\mathbf{i} + [y_B(2-u) + y_C(u-1)]\mathbf{j} + [z_B(2-u) + z_C(u-1)]\mathbf{k}, & u \in [1,2] \end{cases}$$

  We denote it by $L_{A\to B\to C}$.

- A triangle starting at A, going to B, then to C, and coming back to A:

$$[0,3] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3, \tag{1.3}$$

$$\vec{r}(u) = \begin{cases} [x_A(1-u) + x_B u]\mathbf{i} + [y_A(1-u) + y_B u]\mathbf{j} + [z_A(1-u) + z_B u]\mathbf{k}, & u \in [0,1) \\ [x_B(2-u) + x_C(u-1)]\mathbf{i} + [y_B(2-u) + y_C(u-1)]\mathbf{j} + [z_B(2-u) + z_C(u-1)]\mathbf{k}, & u \in [1,2) \\ [x_C(3-u) + x_A(u-2)]\mathbf{i} + [y_C(3-u) + y_A(u-2)]\mathbf{j} + [z_C(3-u) + z_A(u-2)]\mathbf{k}, & u \in [2,3] \end{cases}$$

  We denote it by $L_{A\to B\to C\to A}$.

- A circle parallel to the xOy plane, with radius R and center at $\vec{r}_C = x_C\mathbf{i} + y_C\mathbf{j} + z_C\mathbf{k}$:

$$[0,2\pi] \ni u \mapsto \vec{r}(u) = [x_C + R\cos(u)]\mathbf{i} + [y_C + R\sin(u)]\mathbf{j} + z_C\mathbf{k}. \tag{1.4}$$

  We denote it by $\mathcal{C}_R(\vec{r}_C)$.

It is very important to note that a path is always oriented. It has a starting and an ending point. $L_{A\to B}$ and $L_{B\to A}$ go through the same set of points, but have opposite orientation.

Given any path $\vec{\gamma}$, we can construct the oppositely oriented path $\tilde{\vec{\gamma}}$ by the formula:

$$[a,b] \ni u \mapsto \tilde{\vec{\gamma}}(u) := \vec{\gamma}(a+b-u). \tag{1.5}$$

A path $\vec{\gamma} : [a,b] \to \mathbb{R}^3$ which has the property that $\vec{\gamma}(a) = \vec{\gamma}(b)$ is called a closed path.

1.2 Concatenation of paths

Assume that we have a path $\vec{\gamma}_{A\to B}$ starting at point A and ending at B, and a path $\vec{\gamma}_{B\to C}$ starting at B and ending at C. More precisely:

$$\vec{\gamma}_{A\to B} : [a,b] \to \mathbb{R}^3, \quad \vec{\gamma}_{A\to B}(a) = \vec{r}_A, \quad \vec{\gamma}_{A\to B}(b) = \vec{r}_A,$$

and

$$\vec{\gamma}_{B\to C} : [c,d] \to \mathbb{R}^3, \quad \vec{\gamma}_{B\to C}(c) = \vec{r}_B, \quad \vec{\gamma}_{B\to C}(d) = \vec{r}_C.$$

To concatenate $\vec{\gamma}_{A\to B}$ and $\vec{\gamma}_{B\to C}$ means to define a path $\vec{\gamma}_{A\to B\to C} := \vec{\gamma}_{A\to B} \cup \vec{\gamma}_{B\to C}$ which goes from A to C in the following way:

$$\begin{aligned} &\vec{\gamma}_{A\to B\to C} : \left[\frac{a+c}{2}, \frac{b+d}{2}\right] \in \mathbb{R}^3, \\ &\vec{\gamma}_{A\to B\to C}(u) = \begin{cases} \vec{\gamma}_{A\to B}\left(a + (u - \frac{a+c}{2})\frac{4(b-a)}{b+d-a-c}\right), & u \in [\frac{a+c}{2}, \frac{a+b+c+d}{4}) \\ \vec{\gamma}_{B\to C}\left(c + (u - \frac{a+b+c+d}{4})\frac{4(d-c)}{b+d-a-c}\right), & u \in [\frac{a+b+c+d}{4}, \frac{b+d}{2}] \end{cases} \end{aligned} \tag{1.6}$$

If $a = c$ and $b = d$ then the formulas are simpler:

$$\begin{aligned} &\vec{\gamma}_{A\to B\to C} : [a,b] \in \mathbb{R}^3, \\ &\vec{\gamma}_{A\to B\to C}(u) = \begin{cases} \vec{\gamma}_{A\to B}(a + 2(u-a)), & u \in [a, \frac{a+b}{2}) \\ \vec{\gamma}_{B\to C}\left(a + 2(u - \frac{a+b}{2})\right), & u \in [\frac{a+b}{2}, b] \end{cases} \end{aligned} \tag{1.7}$$

1.3 The velocity vector field

Given a smooth path $\vec{\gamma} : [a,b] \to \mathbb{R}^3$, $\vec{\gamma}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, we can compute its derivative at every point obtaining its velocity vector field:

$$\vec{\gamma}'(u) := x'(u)\mathbf{i} + y'(u)\mathbf{j} + z'(u)\mathbf{k}. \tag{1.8}$$

For example, the velocity field of a straight line $L_{A\to B}$ is given by (see (1.1)):

$$\vec{\gamma}'(u) := (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k} = \vec{r}_B - \vec{r}_A, \tag{1.9}$$

and is constant. In the case of the circle $\mathcal{C}_R(\vec{r}_C)$ we have

$$\vec{\gamma}'(u) := -R\sin(u)\mathbf{i} + R\cos(u)\mathbf{j} + 0\mathbf{k} = -R\sin(u)\mathbf{i} + R\cos(u)\mathbf{j}. \tag{1.10}$$

1.4 The length of a path

Remember that if we have two vectors $\vec{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\vec{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$, their dot product (scalar product) is defined as

$$\vec{r}_1 \cdot \vec{r}_2 = \langle \vec{r}_1, \vec{r}_2 \rangle := x_1x_2 + y_1y_2 + z_1z_2.$$

The length of a vector is

$$||\vec{r}|| := \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{x^2+y^2+z^2}.$$

Consider $\vec{\gamma} : [a,b] \to \mathbb{R}^3$, a smooth path. Its length is defined to be the integral (see also (1.8)):

$$\mathcal{L}(\gamma) := \int_a^b ||\vec{\gamma}'(u)||du = \int_a^b \sqrt{|x'(u)|^2 + |y'(u)|^2 + |z'(u)|^2}du. \tag{1.11}$$

1.5 Path integral of a vector field

Consider a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\vec{F}(\vec{r}) = F_1(\vec{r})\mathbf{i} + F_2(\vec{r})\mathbf{j} + F_3(\vec{r})\mathbf{k}$. Then the path integral of the vector field $\vec{F}$ on the path $\vec{\gamma} : [a, b] \to \mathbb{R}^3$ is defined to be:

$$\int_\gamma \vec{F} \cdot d\vec{\gamma} := \int_a^b \vec{F}(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u)du. \tag{1.12}$$

One can show that the integral does not depend on the way we parameterize the path, as long as one keeps the same orientation. If $\vec{\gamma}$ is a closed path, then the integral is called circulation. Let us see two concrete examples, for the vector field $\vec{F}(\vec{r}) = (x - z)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$.

1. First we compute $\int_{L_{A\to B}} \vec{F} \cdot d\vec{\gamma}$.
   - compute $\vec{\gamma}'(u)$. In this case it equals $\vec{r}_A - \vec{r}_B$;
   - compute the composed function $\vec{F}(\vec{\gamma}(u))$. Here:

     $$\vec{F}(\vec{\gamma}(u)) = [(x_A-z_A)(1-u)+(x_B-z_B)u]\mathbf{i}+[z_A(1-u)+z_Bu]\mathbf{j}+[(y_A+x_A)(1-u)+(y_B+x_B)u]\mathbf{k};$$

   - compute the dot product $\vec{F}(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u)$. Here it gives:

     $$(x_B - x_A)[(x_A - z_A)(1 - u) + (x_B - z_B)u] + (y_B - y_A)[z_A(1 - u) + z_Bu]$$
     $$+(z_B - z_A)[(y_A + x_A)(1 - u) + (y_B + x_B)u].$$

   - integrate from $a = 0$ to $b = 1$. Here we have:

     $$\frac{1}{2}(x_B - x_A)[(x_A - z_A) + (x_B - z_B)] + \frac{1}{2}(y_B - y_A)[z_A + z_B]$$
     $$+\frac{1}{2}(z_B - z_A)[(y_A + x_A) + (y_B + x_B)].$$

2. Second, let us compute the circulation of the same vector field on the circle $\mathcal{C}_R(\vec{r}_C)$.
   - compute $\vec{\gamma}'(u)$. In this case it equals $-R\sin(u)\mathbf{i} + R\cos(u)\mathbf{j}$;
   - compute the composed function $\vec{F}(\vec{\gamma}(u))$. Here:

     $$\vec{F}(\vec{\gamma}(u)) = (x_C + R\cos(u) - z_C)\mathbf{i} + z_C\mathbf{j} + [x_C + y_C + R\cos(u) + R\sin(u)]\mathbf{k};$$

   - compute the dot product $\vec{F}(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u)$. Here it gives:

     $$-R\sin(u)[x_C + R\cos(u) - z_C] + z_CR\cos(u).$$

   - integrate from $a = 0$ to $b = 2\pi$. The result is 0.

1.6 Important properties

We mention two important properties, given without proof. The first one says that if we integrate a vector field on the same path but in the opposite direction, then we get the same numerical value but with the opposite sign. More precisely (see (1.5)):

$$\int_\gamma \vec{F} \cdot d\vec{\gamma} = -\int_{\tilde{\gamma}} \vec{F} \cdot d\vec{\tilde{\gamma}}. \tag{1.13}$$

The second property says that if we integrate a vector field on a concatenated path, then the result is the sum of integrals on individual paths. In detail (see subsection 1.2):

$$\int_{\gamma_{A\to B\to C}} \vec{F} \cdot d\vec{\gamma}_{A\to B\to C} = \int_{\gamma_{A\to B}} \vec{F} \cdot d\vec{\gamma}_{A\to B} + \int_{\gamma_{B\to C}} \vec{F} \cdot d\vec{\gamma}_{B\to C}. \tag{1.14}$$

Now let us consider the situation in which our vector field is the gradient of a given scalar function, that is

$$\vec{F}(\vec{r}) = \nabla V(\vec{r}) = \partial_x V(\vec{r})\mathbf{i} + \partial_y V(\vec{r})\mathbf{j} + \partial_z V(\vec{r})\mathbf{k}.$$

Then we can show that the path integral of $\vec{F}$ between two points A and B is independent of the path linking the two points. Indeed, using the chain rule which gives $\frac{d}{dt}V(\gamma(t)) = \nabla V(\gamma(t)) \cdot \gamma'(t)$ we have:

$$\begin{aligned}\int_\gamma \vec{F} \cdot d\gamma &= \int_a^b \nabla V(\gamma(t)) \cdot \gamma'(t) dt = \int_a^b \left\{ \frac{d}{dt} V(\gamma(t)) \right\} dt = V(\gamma(b)) - V(\gamma(a)) \\ &= V(\vec{r}_B) - V(\vec{r}_A). \end{aligned} \tag{1.15}$$

2 Surface integrals

2.1 General things about surfaces

Any surface in the three dimensional space is parameterized by two real variables. Let $D \subset \mathbb{R}^2$ denote the domain where these parameters live. Mathematically, a surface is the range of a function

$$D \ni (u, v) \mapsto \vec{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \in \mathbb{R}^3.$$

Three important examples:

- Fix two vectors $\vec{r}_A := x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k}$ and $\vec{r}_B := x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}$. The unique plane which contains both vectors can be parameterized as:

$$\mathbb{R}^2 \ni (u, v) \mapsto \vec{r}(u, v) := \vec{r}_A u + \vec{r}_B v = (x_A u + x_B v)\mathbf{i} + (y_A u + y_B v)\mathbf{j} + (z_A u + z_B v)\mathbf{k}. \tag{2.1}$$

  Here $D = \mathbb{R}^2$.

- A sphere with center at $\vec{r}_S = x_S\mathbf{i} + y_S\mathbf{j} + z_S\mathbf{k}$ and radius R:

$$\begin{aligned} &[0, \pi] \times [0, 2\pi] \ni (\theta, \phi) \mapsto \vec{r}(\theta, \phi) \in \mathbb{R}^3, \\ &\vec{r}(\theta, \phi) = \vec{r}_S + R\sin(\theta)\cos(\phi)\mathbf{i} + R\sin(\theta)\sin(\phi)\mathbf{j} + R\cos(\theta)\mathbf{k}. \end{aligned} \tag{2.2}$$

  We denote it by $\partial B_R(\vec{r}_S)$. Here $D = [0, \pi] \times [0, 2\pi]$.

- The disc $\mathcal{D}_R(\vec{r}_C)$ contained in the circle $\mathcal{C}_R(\vec{r}_C)$ (see (1.4)):

$$[0, R] \times [0, 2\pi] \ni (\rho, \phi) \mapsto \vec{r}(\rho, \phi) = [x_C + \rho\cos(\phi)]\mathbf{i} + [y_C + \rho\sin(\phi)]\mathbf{j} + z_C\mathbf{k}; \tag{2.3}$$

2.2 The infinitesimal surface element

Fix a point in D given by (u_0, v_0). If we let (u, v) vary in a small square around (u_0, v_0), then the tip of $\vec{r}(u, v)$ will cover a corresponding small portion of our surface. Assume that $|u - u_0| = \delta u$ and $|v - v_0| = \delta v$ are small. This part of the surface can be approximated by a small portion of the tangent plane touching at $\vec{r}(u_0, v_0)$. Two vectors contained in this tangent plane are

$$\frac{\partial \vec{r}}{\partial u}(u_0, v_0), \quad \text{and} \quad \frac{\partial \vec{r}}{\partial v}(u_0, v_0). \tag{2.4}$$

The area of this surface element will approximately be:

$$d\sigma = \left| \frac{\partial \vec{r}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{r}}{\partial v}(u_0, v_0) \right| \delta u \, \delta v. \tag{2.5}$$

We can also speak about orientation of surfaces. The above small surface element can be associated to the normal vector on the tangent plane, thus we can define a length one vector field

$$\vec{n}(u, v) := \frac{1}{\left| \frac{\partial \vec{r}}{\partial u}(u_0, v_0) \times \frac{\partial \vec{r}}{\partial v}(u_0, v_0) \right|} \frac{\partial \vec{r}}{\partial u}(u, v) \times \frac{\partial \vec{r}}{\partial v}(u, v). \tag{2.6}$$

If we swap u and v we obtain an opposite orientation. In case of closed surfaces, the normal is always taken in such a way that the normal vector points "out of the surface".

2.3 Integration formulas

Now assume that $f(\vec{r})$ is a scalar surface density of a certain physical quantity. Then this quantity is given by the integral:

$$\int_\sigma f(\vec{r})d\sigma := \int_D f(\vec{r}(u,v)) \left|\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)\right| du\, dv. \tag{2.7}$$

If $\vec{F}$ is a vector field, then the flux of $\vec{F}$ through the oriented surface σ is defined to be:

$$\begin{aligned}\int_\sigma \vec{F}(\vec{r}) \cdot d\vec{\sigma} &:= \int_D \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) \left|\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)\right| du\, dv \\ &= \int_D \vec{F}(\vec{r}(u,v)) \cdot \left\{\frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v)\right\} du\, dv.\end{aligned} \tag{2.8}$$

2.4 Two examples

Let us go back to the disk defined in (2.3), and compute its normal vector field and the infinitesimal surface area. First we compute

$$\frac{\partial \vec{r}}{\partial \rho}(\rho,\phi) = \cos(\phi)\mathbf{i} + \sin(\phi)\mathbf{j}, \quad \frac{\partial \vec{r}}{\partial \phi}(\rho,\phi) = -\rho\sin(\phi)\mathbf{i} + \rho\cos(\phi)\mathbf{j}.$$

These two vectors are orthogonal on each other, and moreover (see (2.6)):

$$\vec{n}(\rho,\phi) = \mathbf{k}. \tag{2.9}$$

The surface element is (see (2.5)):

$$d\sigma = \rho\, d\rho d\phi. \tag{2.10}$$

Let us consider a second example, i.e. the sphere in (2.2). In that case:

$$\frac{\partial \vec{r}}{\partial \theta}(\theta,\phi) = R\cos(\theta)\cos(\phi)\mathbf{i} + R\cos(\theta)\sin(\phi)\mathbf{j} - R\sin(\theta)\mathbf{k}, \quad \frac{\partial \vec{r}}{\partial \phi}(\theta,\phi) = -R\sin(\theta)\sin(\phi)\mathbf{i} + R\sin(\theta)\cos(\phi)\mathbf{j}.$$

These two vectors are also orthogonal on each other, and moreover (see (2.6)):

$$\vec{n}(\theta,\phi) = \sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k} = \frac{\vec{r}(\theta,\phi) - \vec{r}_S}{R}. \tag{2.11}$$

The surface element is (see (2.5)):

$$d\sigma = R^2 \sin(\theta)\, d\theta d\phi. \tag{2.12}$$

2.5 Another example

Let us go through a concrete example of surface integration. Consider the vector field $\vec{F}(\vec{r}) = xy\mathbf{i} + xz\mathbf{j}$, and the surface defined by the map:

$$(u,v) \mapsto \vec{r}(u,v) = u\mathbf{i} + v\mathbf{j} + (3u+2v)\mathbf{k}, \quad 0 \le u \le 1, \quad 0 \le v \le 1.$$

Note that this surface can be written as $\vec{r}(u,v) = u\vec{a} + v\vec{b}$ with $\vec{a} = \mathbf{i} + 3\mathbf{k}$ and $\vec{b} = \mathbf{j} + 2\mathbf{k}$. This means that our surface is a parallelogram in the plane generated by the two vectors.

We differentiate and obtain $\frac{\partial \vec{r}}{\partial u}(u,v) = \vec{a}$ and $\frac{\partial \vec{r}}{\partial v}(u,v) = \vec{b}$. Then the normal $\vec{n}(u,v)$ is independent of u and v and is given by:

$$\vec{n}(u,v) = \frac{\vec{a} \times \vec{b}}{||\vec{a} \times \vec{b}||} = -\frac{3}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{1}{\sqrt{14}}\mathbf{k}.$$

Now we compute $\vec{F}(\vec{r}(u,v)) = uv\mathbf{i} + u(3u+2v)\mathbf{j}$ and then the dot product:

$$\vec{F}(\vec{r}(u,v)) \cdot \left\{ \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right\} = \vec{F}(\vec{r}(u,v)) \cdot (\vec{a} \times \vec{b}) = -3uv - 2u(3u+2v) = -6u^2 - 7uv.$$

Finally, according to (2.8) we need to integrate this scalar function on the two dimensional square domain $D = [0,1] \times [0,1]$. We obtain:

$$\int_D (-6u^2 - 7uv)dudv = \int_0^1 \int_0^1 (-6u^2 - 7uv)dudv = -\frac{15}{4}.$$

3 Divergence theorem of Gauss

We will only consider the case of a ball centred at $\vec{r}_S$ and radius R. Remember that such a ball can be represented as:

$$B_R(\vec{r}_S) := \{\vec{r} \in \mathbb{R}^3 : \quad ||\vec{r} - \vec{r}_S|| \leq R\}. \tag{3.1}$$

In cartesian coordinates, a point corresponding to the vector $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ belongs to $B_R(\vec{r}_S)$ if

$$(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2 \leq R^2.$$

The boundary of $B_R(\vec{r}_S)$ is the sphere $\partial B_R(\vec{r}_S)$, which we have already introduced in (2.2).

For a smooth vector field $\vec{F}$, the divergence theorem can be written as:

$$\int_{B_R(\vec{r}_S)} \text{div}\vec{F}(\vec{r})\; dxdydz = \int_{\partial B_R(\vec{r}_S)} \vec{F}(\vec{r}) \cdot \vec{d\sigma}. \tag{3.2}$$

Now let us give a concrete example. Let us assume that the ball is centred at the origin: $\vec{r}_S = \vec{0}$. Consider the vector field $\vec{F}(\vec{r}) = \sqrt{x^2+y^2+z^2}\mathbf{i} + y\mathbf{j}$ and let us compute both integrals in (3.2). We start with the volume integral. The divergence is:

$$\text{div}\vec{F}(\vec{r}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{x}{\sqrt{x^2+y^2+z^2}} + 1.$$

It is convenient to use spherical coordinates, so let us remember some important formulas. We have $r = \sqrt{x^2+y^2+z^2}$ with $0 \leq r \leq R$, and $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, $z = r\cos(\theta)$, with $\theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$. The volume element is $dxdydz = r^2 \sin(\theta) dr d\theta d\phi$. In spherical coordinates: $\text{div}\vec{F}(\vec{r}) = \sin(\theta)\cos(\phi) + 1$. Then we can write:

$$\begin{aligned}\int_{B_R(\vec{0})} \text{div}\vec{F}(\vec{r})\; dxdydz &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\; \text{div}\vec{F}(\vec{r})\; r^2 \sin(\theta) \\ &= \frac{R^3}{3} \int_0^\pi d\theta \int_0^{2\pi} d\phi\; [\sin^2(\theta)\cos(\phi) + \sin(\theta)] = \frac{4\pi R^3}{3}. \end{aligned} \tag{3.3}$$

Now let us compute the surface integral in (3.2) and show that it gives the same result. We have computed the normal to the surface $\vec{n}(\theta,\phi) = \sin(\theta)\cos(\phi)\mathbf{i} + \sin(\theta)\sin(\phi)\mathbf{j} + \cos(\theta)\mathbf{k}$ in (2.11). We can also express $\vec{F}(\vec{r})$ using spherical coordinates as $\vec{F}(\vec{r}) = r\mathbf{i} + r\sin(\theta)\sin(\phi)\mathbf{j}$. If $\vec{r}$ lies on the sphere we have to fix $r = R$. Then we have (see also (2.12)):

$$\vec{F}(\vec{r}(\theta,\phi)) \cdot \vec{d\sigma} = \vec{F}(\vec{r}(\theta,\phi)) \cdot \vec{n}(\theta,\phi) d\sigma = R^3[\sin(\theta)\cos(\phi) + \sin^2(\theta)\sin^2(\phi)]\sin(\theta) d\theta d\phi. \tag{3.4}$$

Now we have to integrate over θ and ϕ. It is useful to know the following two identities: $\sin^2(\phi) = \left(\frac{\phi}{2} - \frac{\sin(2\phi)}{4}\right)'$ and $\sin^3(\theta) = \left(-\cos(\theta) + \frac{\cos^3(\theta)}{3}\right)'$. Thus:

$$\int_{\partial B_R(\vec{0})} \vec{F}(\vec{r}) \cdot \vec{d\sigma} = R^3 \int_0^\pi \sin^3(\theta) d\theta \int_0^{2\pi} \sin^2(\phi) d\phi = \frac{4\pi R^3}{3}.$$

4 Stokes' theorem

We will only consider the disk $\mathcal{D}_R(\vec{r}_C)$, which in cartesian coordinates is characterized by:

$$z = z_C, \quad (x - x_C)^2 + (y - y_C)^2 \leq R^2.$$

For simplicity, let us assume that $\vec{r}_C = \vec{0}$. In this case, the disk is parameterized by $\vec{r}(\rho, \phi) = \rho\cos(\phi)\mathbf{i} + \rho\sin(\phi)\mathbf{j}$ where $0 \leq \rho \leq R$ and $0 \leq \phi \leq 2\pi$ (see (2.3)). From (2.9) we know that the normal is $\vec{n}(\rho, \phi) = \mathbf{k}$. The infinitesimal surface element is (see (2.10)) $d\sigma = \rho\ d\rho d\phi$.

The boundary of $\mathcal{D}_R(\vec{0})$ is the oriented circle $\mathcal{C}_R(\vec{0})$ which is parameterized as (see (1.4)):

$$\vec{\gamma}_R(u) = R\cos(u)\mathbf{i} + R\sin(u)\mathbf{j}, \quad u \in [0, 2\pi].$$

Consider $\vec{F}$ a smooth vector field given by $\vec{F}(\vec{r}) = F_x(\vec{r})\mathbf{i} + F_y(\vec{r})\mathbf{j} + F_z(\vec{r})\mathbf{k}$. Then Stokes' theorem states the following equality:

$$\int_{\mathcal{D}_R(\vec{0})} (\mathrm{curl}\vec{F}) \cdot d\vec{\sigma} = \int_{\gamma_R} \vec{F} \cdot d\vec{\gamma}_R \tag{4.1}$$

Consider the vector field $\vec{F}(\vec{r}) = (x + y + z)\mathbf{i} + x^2\mathbf{j}$ and let us compute both sides of the above equality. We start with the surface integral.

First we need to compute the curl. Since $d\vec{\sigma}$ is always parallel with $\mathbf{k}$, it suffices to only compute the $\mathbf{k}$ component of the curl. This component is

$$(curl\vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2x - 1.$$

Then

$$(\mathrm{curl}\vec{F}) \cdot d\vec{\sigma} = (2x - 1)\rho\ d\rho d\phi = [2\rho\cos(\phi) - 1)]\rho\ d\rho d\phi,$$

and

$$\int_{\mathcal{D}_R(\vec{0})} (\mathrm{curl}\vec{F}) \cdot d\vec{\sigma} = \int_0^R d\rho \int_0^{2\pi} d\phi\ [2\rho\cos(\phi) - 1)]\rho = -\pi R^2.$$

Now let us compute the path integral of $\vec{F}$ over $\vec{\gamma}_R$. We first compute $\vec{\gamma}_R'(u) = -R\sin(u)\mathbf{i} + R\cos(u)\mathbf{j}$. Then we have:

$$\vec{F} \cdot d\vec{\gamma}_R = \vec{F}(\vec{\gamma}_R(u)) \cdot \vec{\gamma}_R'(u)du = \{-R^2[\cos(u) + \sin(u)]\sin(u) + R^3\cos^3(u)\}du.$$

We have to integrate from 0 to 2π. It is useful to know the identities $\cos(u)\sin(u) = \left(\frac{\sin^2(u)}{2}\right)'$, $\sin^2(u) = \left(\frac{u}{2} - \frac{\sin(2u)}{4}\right)'$ and $\cos^3(u) = \left(\sin(u) - \frac{\sin^3(u)}{3}\right)'$. Inserting the limits we obtain:

$$\int_{\gamma_R} \vec{F} \cdot d\vec{\gamma}_R = -\pi R^2.$$

5 A crash course in electro and magnetostatics

A stationary electric field generated by some electric charges with a known volume density $\rho(\vec{r})$ can be modeled by a vector field $\vec{E}(\vec{r})$ which obeys two (Maxwell) equations: $\nabla \cdot \vec{E}(\vec{r}) = \rho(\vec{r})$ and $\nabla \times \vec{E}(\vec{r}) = \vec{0}$. A stationary magnetic field generated by a known current density $\vec{j}(\vec{r})$ can be modeled by a vector field $\vec{B}(\vec{r})$ which obeys other two Maxwell equations: $\nabla \cdot \vec{B}(\vec{r}) = 0$ and $\nabla \times \vec{B}(\vec{r}) = \vec{j}(\vec{r})$. The main question is how to recover $\vec{E}$ and $\vec{B}$ from the knowledge of ρ and $\vec{j}$.

From a mathematical point of view, this problem can be formulated in a more general way. Assume that an unknown vector field $\vec{F}(\vec{r}) = F_x(\vec{r})\mathbf{i} + F_y(\vec{r})\mathbf{j} + F_z(\vec{r})\mathbf{k}$ obeys the equations

$$\nabla \cdot \vec{F}(\vec{r}) = f(\vec{r}), \quad \nabla \times \vec{F}(\vec{r}) = \vec{g}(\vec{r}), \tag{5.1}$$

where f and $\vec{g}$ are some known smooth functions, and localized in a finite region of the space near the origin. Can we find such a vector field $\vec{F}$?

The answer is yes, and the solution is even unique if we are only interested in an $\vec{F}$ which "goes to zero at infinity". Let us construct this solution.

First, let us define the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ which formally can be seen as $\nabla \cdot \nabla = \nabla^2$. The Laplace operator acts on a scalar field ϕ in the following way:

$$\Delta\phi(\vec{r}) := \frac{\partial^2 \phi}{\partial x^2}(\vec{r}) + \frac{\partial^2 \phi}{\partial y^2}(\vec{r}) + \frac{\partial^2 \phi}{\partial z^2}(\vec{r}),$$

while on a vector field $\vec{F}$ it acts separately on each component:

$$\Delta\vec{F}(\vec{r}) := (\Delta F_x)(\vec{r})\mathbf{i} + (\Delta F_y)(\vec{r})\mathbf{j} + (\Delta F_z)(\vec{r})\mathbf{k}.$$

Let us note an important equality given without proof, but which you may verify starting by the definitions of curl and div:

$$\nabla \times (\nabla \times \vec{F}) = -\Delta\vec{F} + \nabla(\nabla \cdot \vec{F}), \tag{5.2}$$

or in the other notation:

$$\mathrm{curl}(\mathrm{curl}\vec{F}) = -\Delta\vec{F} + \mathrm{grad}(\mathrm{div}\vec{F}). \tag{5.3}$$

Introducing the input from (5.1) in the above equation we obtain:

$$-\Delta\vec{F} = \mathrm{curl}\ \vec{g} - \mathrm{grad}\ f = \nabla \times \vec{g} - \nabla f. \tag{5.4}$$

This is nothing but the Poisson equation which in general looks like $-\Delta\vec{F}(\vec{r}) = \vec{H}(\vec{r})$, where $\vec{H}$ is a known vector field. In three dimensions, the unique solution which goes to zero at infinity is given in terms of the Green function $G_0(\vec{r}) := \frac{1}{4\pi|\vec{r}|}$ in the following way:

$$\vec{F}(\vec{r}) = \int_{\mathbb{R}^3} G_0(\vec{r} - \vec{r}')\vec{H}(\vec{r}')d\vec{r}' = \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}\vec{H}(\vec{r}')d\vec{r}'.$$

Therefore, in our case we have:

$$\vec{F}(\vec{r}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}[\nabla \times \vec{g}(\vec{r}') - \nabla f(\vec{r}')]d\vec{r}'.$$

It turns out that the ∇ operator commutes with the application of the Green function, and we can write the solution as:

$$\vec{F}(\vec{r}) = \nabla \times \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}\vec{g}(\vec{r}')d\vec{r}' - \nabla \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}f(\vec{r}')d\vec{r}'. \tag{5.5}$$

If we introduce the notation:

$$\vec{A}(\vec{r}) := \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}\vec{g}(\vec{r}')d\vec{r}', \quad V(\vec{r}) := \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}f(\vec{r}')d\vec{r}', \tag{5.6}$$

then we can write the solution as

$$\vec{F}(\vec{r}) = \mathrm{curl}\ \vec{A}(\vec{r}) - \mathrm{grad}\ V(\vec{r}).$$

- In the case of the electric field we have curl$\vec{E} = \vec{0}$ and div$\vec{E} = \rho$, thus

$$\vec{E}(\vec{r}) = -\text{grad}\, V(\vec{r}), \quad V(\vec{r}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}\rho(\vec{r}')d\vec{r}',$$

where V is called the electric scalar potential.

- In the case of the magnetic field we have curl$\vec{B} = \vec{j}$ and div$\vec{B} = 0$, thus

$$\vec{B}(\vec{r}) = \text{curl}\, \vec{A}(\vec{r}), \quad \vec{A}(\vec{r}) := \int_{\mathbb{R}^3} \frac{1}{4\pi|\vec{r} - \vec{r}'|}\vec{j}(\vec{r}')d\vec{r}',$$

where $\vec{A}$ is called the magnetic vector potential.

6 Examples of exam exercices

Exercise 6.1. *Using the Laplace transform, solve the equation $y''(t) - y(t) = 3t - 2$ knowing the initial conditions $y(1) = 0$ and $y'(1) = 1$.*

Solution. Because the initial condition is given at $t_0 = 1$ and not at 0, we need to use the same method as in Example 6, page 232 in Kreyszig. We introduce the new variable $\tilde{t} = t - 1$ so that the old variable becomes $t = \tilde{t} + 1$. We define $\tilde{y}(\tilde{t}) = y(\tilde{t} + 1)$, and see that using the chain rule we can write:

$$\tilde{y}'(\tilde{t}) = y'(\tilde{t} + 1), \quad \tilde{y}''(\tilde{t}) = y''(\tilde{t} + 1),$$

and the new initial conditions are $\tilde{y}(0) = 0$ and $\tilde{y}'(0) = 1$.

The idea is to write an equation for $\tilde{y}$. Introducing $t = \tilde{t}+1$ in the original equation we obtain $y''(\tilde{t} + 1) - y(\tilde{t} + 1) = 3\tilde{t} + 1$, or equivalently

$$\tilde{y}''(\tilde{t}) - \tilde{y}(\tilde{t}) = 3\tilde{t} + 1.$$

Now we can perform the Laplace transform in both sides of the above equation. Denoting by $\tilde{Y}(s)$ the Laplace tranform of $\tilde{y}$ and using the rules given in Theorem 1, page 228 in Kreyszig, we get:

$$s^2\tilde{Y}(s) - s\tilde{y}(0) - \tilde{y}'(0) - \tilde{Y}(s) = (s^2 - 1)\tilde{Y}(s) - 1 = \frac{3}{s^2} + \frac{1}{s}.$$

This gives:

$$\tilde{Y}(s) = \frac{1}{s^2 - 1} + \frac{3}{s^2(s^2 - 1)} + \frac{1}{s(s^2 - 1)}.$$

Using formula 15 on page 265 in Kreyszig we obtain that the inverse Laplace transform of $1/(s^2-1)$ is $\sinh(\tilde{t})$. Now using Theorem 3 on page 229 in Kreyszig we obtain that the inverse Laplace transform of $\frac{1}{s(s^2-1)}$ is $\cosh(\tilde{t})-1$, and using the same Theorem again the inverse Laplace transform of $\frac{1}{s^2(s^2-1)}$ is $\sinh(\tilde{t}) - \tilde{t}$. Thus:

$$\tilde{y}(\tilde{t}) = 4\sinh(\tilde{t}) - 3\tilde{t} + \cosh(\tilde{t}) - 1 = \frac{5}{2}e^{\tilde{t}} - \frac{3}{2}e^{-\tilde{t}} - 3\tilde{t} - 1.$$

Then the solution is:

$$y(t) = \tilde{y}(t - 1) = \frac{5}{2}e^{t-1} - \frac{3}{2}e^{-t+1} - 3t + 2.$$

Exercise 6.2. *Solve the system of equations $y_1'(t) = y_2(t) + 3$ and $y_2'(t) = -4y_1(t) + e^{-t}$, with $y_1(0) = 2$ and $y_2(0) = 1$.*

Solution. Denote by $Y_1(s)$ and $Y_2(s)$ the Laplace transforms of y_1 and y_2 respectively. Take the Laplace transform of both equations, using the first rule given in Theorem 1, page 228 in Kreyszig. We obtain two algebraic equations

$$sY_1(s) - 2 = Y_2(s) + \frac{3}{s}, \qquad sY_2(s) - 1 = -4Y_1(s) + \frac{1}{s+1}.$$

One can determine Y_1 and Y_2 by various methods. They are given by:

$$Y_1(s) = 2\frac{s}{s^2+4} + \frac{4}{s^2+4} + \frac{1}{(s^2+4)(s+1)}, \; Y_2(s) = -\frac{7}{s^2+4} - \frac{12}{s(s^2+4)} + \frac{s}{s^2+4} - \frac{1}{(s^2+4)(s+1)}.$$

The next step is to take the inverse Laplace transform and find $y_1(t)$ and $y_2(t)$. Using formula (13) on page 265 in Kreyszig we obtain that the inverse Laplace transform of $\frac{1}{s^2+4}$ is $\frac{\sin(2t)}{2}$, from formula (14) on the same page we have that the inverse Laplace transform of $\frac{s}{s^2+4}$ is $\cos(2t)$, while formula (19) says that the inverse Laplace transform of $\frac{1}{s(s^2+4)}$ is $\frac{1-\cos(2t)}{4}$. But $\frac{1}{(s^2+4)(s+1)}$ has to be decomposed into simpler fractions, which we do now.

We search for three constants A, B and C such that the following equality to hold for all s:

$$\frac{1}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}.$$

This is equivalent with:

$$1 = (As+B)(s+1) + C(s^2+4) = (A+C)s^2 + (A+B)s + B + 4C.$$

This equality can hold only if $A+C = 0$, $A+B = 0$ and $B+4C = 1$. This gives $B = C = 1/5 = -A$, thus:

$$\frac{1}{(s^2+4)(s+1)} = -\frac{1}{5}\frac{s}{s^2+4} + \frac{1}{5}\frac{1}{s^2+4} + \frac{1}{5}\frac{1}{s+1}.$$

Introducing this formula in the expression of $Y_1(s)$ we obtain:

$$Y_1(s) = \frac{9}{5}\frac{s}{s^2+4} + \frac{21}{5}\frac{1}{s^2+4} + \frac{1}{5}\frac{1}{s+1},$$

and

$$Y_2(s) = -\frac{36}{5}\frac{1}{s^2+4} - \frac{12}{s(s^2+4)} + \frac{6}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{1}{s+1}$$

Now Y_1 is written as a sum of simpler function whose inverse Laplace transform can be found in Kreyszig. The first two terms appeared already before, while the third term has the inverse Laplace transform determined by formula (7) on page 265 in Kreyszig. We get:

$$y_1(t) = \frac{9\cos(2t)}{5} + \frac{21\sin(2t)}{10} + \frac{e^{-t}}{5}$$

and

$$y_2(t) = -\frac{18\sin(2t)}{5} - 3(1-\cos(2t)) + \frac{6\cos(2t)}{5} - \frac{e^{-t}}{5} = -\frac{18\sin(2t)}{5} + \frac{21\cos(2t)}{5} - 3 - \frac{e^{-t}}{5}.$$

Exercise 6.3. *Consider the vector field $\vec{F}(\vec{r}) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ and the surface σ defined by the mapping:*

$$[0,1] \times [0,2\pi] \ni (\rho,\phi) \mapsto \vec{r}(\rho,\phi) = 4\mathbf{i} + [1+\rho\cos(\phi)]\mathbf{j} + [2+\rho\sin(\phi)]\mathbf{k}.$$

(i). Show that σ is the a disk parallel with the plane yOz, with center at $\vec{r}_C = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and of radius $R = 1$.

(ii). Compute $\nabla \cdot \vec{F}(\vec{r})$ and $\nabla \times \vec{F}(\vec{r})$.

(iii). Compute the surface integral $\int_\sigma \vec{F} \cdot \vec{d\sigma}$.

Solution.

(i). If $\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a vector which belongs to σ, then we must have $x = 4$, $y - 1 = \rho\cos(\phi)$ and $z - 2 = \rho\sin(\phi)$. Thus $(y-1)^2 + (z-2)^2 = \rho^2 \le 1$, which is the equation of a disk of radius $R = 1$ in the plane yOz with center at $y_C = 1$ and $z_C = 2$. Since $x = 4$ is constant, the disk is situated in a plane parallel with yOz, given by the equation $x = 4$.

(ii). Denoting by $\vec{F}(\vec{r}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, we have $F_x(\vec{r}) = y^2$, $F_y(\vec{r}) = z^2$ and $F_z(\vec{r}) = x^2$. Then $\nabla \cdot \vec{F}(\vec{r}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$. Moreover,

$$\nabla \times \vec{F}(\vec{r}) = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\mathbf{i} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\mathbf{j} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\mathbf{k} = -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}.$$

(iii). We use formula (2.8) in these notes. We need to compute the vectors:

$$\frac{\partial \vec{r}}{\partial \rho}(\rho,\phi) = \cos(\phi)\mathbf{j} + \sin(\phi)\mathbf{k}, \quad \frac{\partial \vec{r}}{\partial \phi}(\rho,\phi) = -\rho\sin(\phi)\mathbf{j} + \rho\cos(\phi)\mathbf{k}, \quad \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi} = \rho\mathbf{i}.$$

In particular, the normal to the surface is

$$\vec{n}(\rho,\phi) = \frac{\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi}}{\left\| \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi} \right\|} = \mathbf{i},$$

which confirms that the surface is contained in a plane parallel with yOz.

Now

$$\vec{F}(\vec{r}(\rho,\phi)) \cdot \left\{ \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi} \right\}(\rho,\phi) = \rho F_x(\vec{r}(\rho,\phi)) = \rho[1 + \rho\cos(\phi)]^2 = \rho + 2\rho^2\cos(\phi) + \rho^3\cos^2(\phi).$$

According to (2.8) we have:

$$\int_\sigma \vec{F} \cdot d\vec{\sigma} = \int_0^1 d\rho \int_0^{2\pi} d\phi[\rho + 2\rho^2\cos(\phi) + \rho^3\cos^2(\phi)] = 5\pi/4.$$

Exercise 6.4. *Consider the vector field $\vec{F}(\vec{r}) = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$. Let $\vec{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$ be a fixed vector, and let $\vec{\gamma} : [0,1] \to \mathbb{R}^3$ be the segment joining the origin $\vec{0}$ with $\vec{R}$, given by the following formula:*

$$\vec{\gamma}(u) = u\vec{R} = uR_x\mathbf{i} + uR_y\mathbf{j} + uR_z\mathbf{k}, \quad 0 \le u \le 1.$$

(i). Compute the path integral $\int_\gamma \vec{F} \cdot d\vec{\gamma}$, and denote it with $\phi(\vec{R})$. Show that $\nabla\phi(\vec{r}) = \vec{F}(\vec{r})$.

(ii). Compute $\nabla \cdot \vec{F}(\vec{r})$ and $\nabla \times \vec{F}(\vec{r})$.

Solution.

(i). We have $\vec{\gamma}'(u) = \vec{R}$, $\vec{F}(\vec{\gamma}(u)) = u^5R_xR_y^2R_z^2\mathbf{i} + u^5R_x^2R_yR_z^2\mathbf{j} + u^5R_x^2R_y^2R_z\mathbf{k}$ and

$$\vec{F}(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u) = 3u^5R_x^2R_y^2R_z^2.$$

According to formula (1.12) we have:

$$\int_\gamma \vec{F} \cdot d\vec{\gamma} = \int_0^1 \vec{F}(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u)du = \frac{R_x^2R_y^2R_z^2}{2}.$$

Thus $\phi(\vec{r}) = \frac{x^2y^2z^2}{2}$ and $\nabla\phi(\vec{r}) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = xy^2z^2\mathbf{i} + x^2yz^2\mathbf{j} + x^2y^2z\mathbf{k}$.

(ii). $\nabla \cdot \vec{F}(\vec{r}) = y^2z^2 + x^2z^2 + x^2y^2$ and $\nabla \times \vec{F}(\vec{r}) = \vec{0}$.

Exercise 6.5. *Consider the vector field $\vec{F}(\vec{r}) = -z^2\mathbf{j} + y^2\mathbf{k}$. Let σ be the surface defined by the map*

$$\vec{r}(u,v) = (2u+v)\mathbf{i} + u\mathbf{j} + v^2\mathbf{k}, \quad u^2+v^2 \le 1$$

and let $\vec{\gamma} : [0, 2\pi] \to \mathbb{R}^3$ given by

$$\vec{\gamma}(t) = [2\cos(t) + \sin(t)]\mathbf{i} + \cos(t)\mathbf{j} + \sin^2(t)\mathbf{k}.$$

(i) Compute the integral $\int_\sigma (\nabla \times \vec{F}) \cdot d\vec{\sigma}$.
(ii) Compute $\int_\gamma \vec{F} \cdot d\vec{\gamma}$.

Solution.
(i). We have $\frac{\partial \vec{r}}{\partial u}(u,v) = 2\mathbf{i} + \mathbf{j}$, $\frac{\partial \vec{r}}{\partial v}(u,v) = \mathbf{i} + 2v\mathbf{k}$, $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = 2v\mathbf{i} - 4v\mathbf{j} - \mathbf{k}$ and the normal to the surface is

$$\vec{n}(u,v) = \frac{2v}{\sqrt{20v^2+1}}\mathbf{i} - \frac{4v}{\sqrt{20v^2+1}} - \frac{1}{\sqrt{20v^2+1}}\mathbf{k}.$$

The curl of $\vec{F}$ is $\nabla \times \vec{F}(\vec{r}) = (2y+2z)\mathbf{i}$, thus $\nabla \times \vec{F}(\vec{r}(u,v)) = (2u+2v^2)\mathbf{i}$. It follows that:

$$\nabla \times \vec{F}(\vec{r}(u,v)) \cdot \left\{ \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\} = 4(uv + v^3).$$

The domain D of the variables u and v is the unit disk defined by $u^2+v^2 \le 1$, with center at the origin. Thus we have:

$$\int_\sigma (\nabla \times \vec{F}) \cdot d\vec{\sigma} = \int_D \nabla \times \vec{F}(\vec{r}(u,v)) \cdot \left\{ \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\} dudv = 4\int_D (uv+v^3)dudv.$$

In order to compute the last integral we introduce polar coordinates $u = \rho\cos(\phi)$ and $v = \rho\sin(\phi)$ with $0 \le \rho \le 1$ and $0 \le \phi \le 2\pi$ which gives:

$$4\int_D (uv+v^3)dudv = 4\int_0^1 \int_0^{2\pi} [\rho^2\cos(\phi)\sin(\phi) + \rho^3\sin^3(\phi)]\rho d\rho d\phi = 0.$$

(ii). We have $\vec{\gamma}'(t) = [-2\sin(t) + \cos(t)]\mathbf{i} - \sin(t)\mathbf{j} + 2\sin(t)\cos(t)\mathbf{k}$, $\vec{F}(\vec{\gamma}(t)) = -\sin^4(t)\mathbf{j} + \cos^2(t)\mathbf{k}$ and

$$\vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) = \sin^5(t) + 2\sin(t)\cos^3(t).$$

In conclusion

$$\int_\gamma \vec{F} \cdot d\vec{\gamma} = \int_0^{2\pi} \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)dt = \int_0^{2\pi} [\sin^5(t) + 2\sin(t)\cos^3(t)]dt = 0.$$

Exercise 6.6. *Consider the equation $y''(t) + y'(t) + 2y(t) = r(t)$ where the driving force $r(t)$ is 2π-periodic and is defined by*

$$r(t) = \begin{cases} t + \pi/2, & -\pi < t < 0 \\ -t + \pi/2, & 0 \le t \le \pi \end{cases}.$$

Find the steady-state solution.

Solution. The driving force $r(t)$ is an even function with a cosine-Fourier decomposition given by $r(t) = a_0 + \sum_{n \ge 1} a_n \cos(nt)$, where (see Theorem 1 on page 491 in Kreyszig where we put $L = \pi$)

$$a_0 = \frac{1}{\pi}\int_0^\pi (-t+\pi/2)dt = 0, \quad a_n = \frac{2}{\pi}\int_0^\pi (-t+\pi/2)\cos(nt)dt, \quad n = 1, 2, ...$$

Computing the integral we obtain:

$$a_0 = 0, \quad a_n = \frac{2[1-(-1)^n]}{n^2\pi}, \quad n = 1, 2, ...$$

This formula says that only odd n's give nonzero contributions. The steady state solution $y_S(t)$ can be expressed as a sum $y_S(t) = \sum_{n\geq 0} y_n(t)$ where y_n is a solution of the equation

$$y_n''(t) + y_n'(t) + 2y_n(t) = a_n \cos(nt). \tag{6.1}$$

The signal $y_n(t)$ is of the form $y_n(t) = A_n \cos(nt) + B_n \sin(nt)$ with $n = 1, 3, 5, ...$, and

$$y'(t) = -nA_n \sin(nt) + nB_n \cos(nt), \quad y''(t) = -n^2 A_n \cos(nt) - n^2 B_n \sin(nt).$$

Inserting this in the equation (6.1) we have:

$$(-n^2 A_n + nB_n + 2A_n)\cos(nt) + (-n^2 B_n - nA_n + 2B_n)\sin(nt) = a_n \cos(nt).$$

This gives two equations for A_n and B_n:

$$-n^2 A_n + nB_n + 2A_n = a_n, \quad -n^2 B_n - nA_n + 2B_n = 0, \quad n = 1, 3, 5, ...$$

The solutions of this system of equations are $A_n = -\frac{n^2-2}{n^4-3n^2+4} a_n$ and $B_n = \frac{n}{n^4-3n^2+4} a_n$.